Q. A firm can produce output with 2 alternative technologies given by $q = \min\left\{\frac{K}{3}, \frac{L}{2}\right\}$ and $q = \min\left\{\frac{K}{2}, \frac{L}{3}\right\}$. The marginal cost of production is 20 with both the technologies. Find the expansion path for the prodn fn: $q = K^{2/3}L^{1/3}$.

Prodn fn 1: $q = \min\left\{\frac{K}{3}, \frac{L}{2}\right\}$

$\therefore$ At opt: $\frac{K}{3} = \frac{L}{2} = q \Rightarrow K^* = 3q,\ L^* = 2q$

$\therefore\ C_1 = wL^* + rK^* = (2w + 3r)q$.

Prodn fn 2: $q = \min\left\{\frac{K}{2}, \frac{L}{3}\right\}$

$\therefore$ At opt: $\frac{K}{2} = \frac{L}{3} = q \Rightarrow K^* = 2q,\ L^* = 3q$

$\therefore\ C_2 = wL^* + rK^* = (3w + 2r)q$

$\therefore\ MC_1 = MC_2 = 20$.

$MC_1 = \frac{\partial C_1}{\partial q} = (2w + 3r) \qquad MC_2 = \frac{\partial C_2}{\partial q} = (3w + 2r)$

$\therefore$ Given: $\left.\begin{aligned} 2w + 3r &= 20. \\ 3w + 2r &= 20 \end{aligned}\right\} \Rightarrow$ solve for w, r

$w = r = 4$

Expansion path: $\frac{MP_L}{MP_K} = \frac{w}{r}$, $\quad q = K^{2/3}L^{1/3}$

$MP_L = \frac{\partial q}{\partial L} = \frac{1}{3}K^{2/3}L^{-2/3}$.

$MP_K = \frac{\partial q}{\partial K} = \frac{2}{3}K^{-1/3}L^{1/3}$.

$\therefore\ \frac{MP_L}{MP_K} = \frac{\frac{1}{\cancel{3}}K^{2/3}L^{-2/3}}{\frac{2}{\cancel{3}}\cdot K^{-1/3}L^{1/3}} = \frac{1}{2}\cdot\left(\frac{K}{L}\right)$

$\therefore$ Expansion path: $\frac{MP_L}{MP_K} = \frac{w}{r} = 1 \Rightarrow \frac{1}{2}\cdot\left(\frac{K}{L}\right) = 1$

$\therefore$ Expansion path: $\frac{MP_L}{MP_K} = \frac{w}{r} = 1 \Rightarrow \frac{1}{2}\left(\frac{K}{L}\right) = 1$

$\Rightarrow \boxed{K = 2L}$

Concave & Convex Fns.

Eg: $y = f(x)$ -- [univariate]

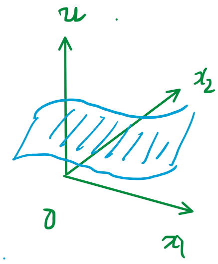
Concave: $\frac{d^2y}{dx^2} < 0$

Convex: $\frac{d^2y}{dx^2} > 0$

$z = f(x, y)$ --- [multivariate]

Second order double derivatives:

$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$.

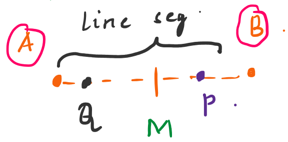


Another criteria for checking concavity & convexity:

Eg: Consider 2 pts $A(x_1, y_1)$ $B(x_2, y_2)$ on 2-D plane.

Mid pt of $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \frac{1}{2}A + \frac{1}{2}B$.



$P = \frac{1}{3}A + \frac{2}{3}B$.

$Q = \frac{3}{4}A + \frac{1}{4}B$.

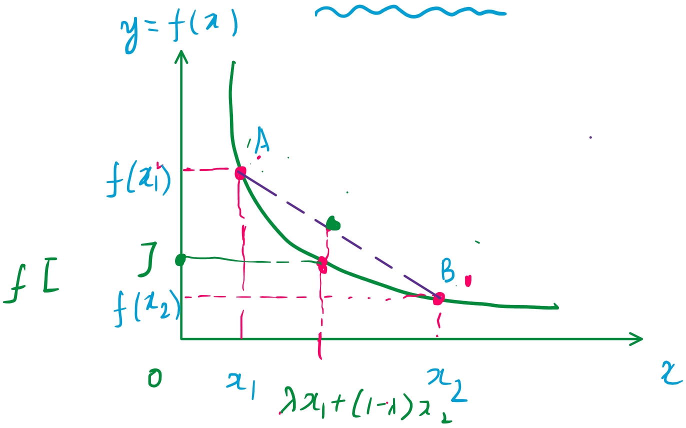
Convex combination:

Suppose we have 2 pts on the 2-D plane: $x(x_1, x_2)$ and $y(y_1, y_2)$

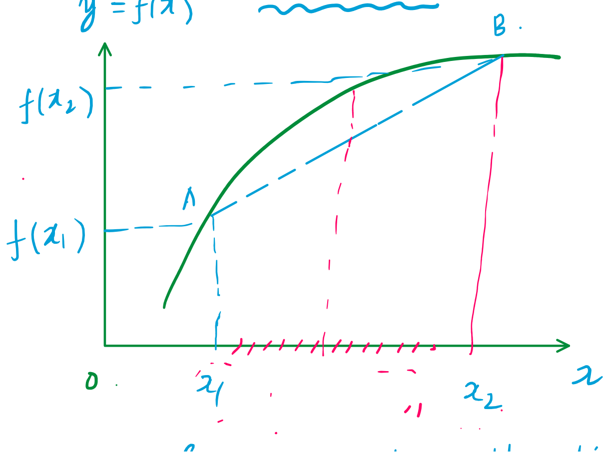
Convex combination of $x, y = \lambda x + (1-\lambda)y$, $0 \leq \lambda \leq 1$.

Graphically convex combination traces out the line segment b/w the 2 pts x & y.

Convex Fn



Concave Fn



$0 \quad x_1 \quad \lambda x_1 + (1-\lambda) x_2 \quad x_2 \qquad x$ $\qquad\qquad 0 \quad x_1 \qquad x_2 \quad x$

Line is above the curve. $\qquad\qquad$ Curve is above the line.

Eqn of line: $\lambda \cdot f(x_1) + (1-\lambda) \cdot f(x_2)$, $0 \leq \lambda \leq 1$.

Eqn of Curve: $f[\lambda \cdot x_1 + (1-\lambda) x_2]$.

Convex: Line above curve:

$$\lambda f(x_1) + (1-\lambda) f(x_2) > f[\lambda x_1 + (1-\lambda) x_2]$$

Concave: Curve above line:

$$f[\lambda x_1 + (1-\lambda) x_2] > \lambda f(x_1) + (1-\lambda) f(x_2)$$

Q. Show that the cost fn is concave in input prices.

Input prices: (w, r)

Factors of production: (L, K).

Cost fn: $C = w L^*(w, r, q) + r \cdot K^*(w, r, q)$

$$= c(w, r, q) \text{ --- (cost fn).}$$

Consider 2 combinations of factor prices:

$(w_1, r_1) = f_1$ $\qquad\qquad$ $(w_2, r_2) = f_2$.

$\hookrightarrow L_1, K_1 \Rightarrow c_1$ $\qquad\qquad$ $\hookrightarrow L_2, K_2 \Rightarrow c_2$

Consider a convex combination of (w_1, r_1) (w_2, r_2)

$$f_3 = (w_3, r_3) = \lambda (w_1, r_1) + (1-\lambda)(w_2, r_2)$$

$$\hookrightarrow L_3, K_3 \Rightarrow c_3.$$

Expression of $c_3 = w_3 \cdot L_3 + r_3 K_3$.

$$= [\lambda w_1 + (1-\lambda) w_2] L_3 + [\lambda r_1 + (1-\lambda) r_2] K_3 .$$

$$= \lambda \underbrace{[w_1 L_3 + r_1 K_3]}_{\text{cost where } (w_1, r_1) \text{ \& chosen combination } (L_3, K_3)} + (1-\lambda) \underbrace{[w_2 L_3 + r_2 K_3]}_{\text{cost where } (w_2, r_2) \text{ \& chosen combination } (L_3, K_3)}$$

$w_1 L_3 + r_1 K_3 > C_1 .$ $\qquad$ $w_2 L_3 + r_2 K_3 > C_2 .$

$$\therefore \underset{\substack{\downarrow \\ \text{Curve}}}{C_3} > \underbrace{\lambda \cdot C_1 + (1-\lambda) C_2}_{\text{Line}} . \Rightarrow \text{concave.}$$