08 January 2024 07:06 PM

$$\checkmark\ \Sigma x_{2i} y_i - \bar{Y} \Sigma x_{2i} = \hat{\beta}_2 \left(\Sigma x_{2i}^2 - \bar{x}_2 \Sigma x_{2i}\right)$$

$$+ \hat{\beta}_3 \left(\Sigma x_{2i} x_{3i} - \bar{x}_3 \Sigma x_{2i}\right)$$

Using the results : $\Sigma x_i y_i - \bar{Y} \Sigma x_i$

$$= \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

$$= \Sigma x_i y_i$$

and $\Sigma x_i^2 - \bar{x} \Sigma x_i = \Sigma (x_i - \bar{x})^2 = \Sigma x_i^2$

$$\Sigma x_2 y = \hat{\beta}_2 \Sigma x_2^2 + \hat{\beta}_3 \Sigma x_2 x_3 \longrightarrow (1)$$

$$\bar{Y} - \hat{\beta}_2 \bar{x}_2 - \hat{\beta}_3 \bar{x}_3 = \hat{\beta}_1$$

$$\frac{\partial \Sigma e^2}{\partial \hat{\beta}_3} = -2 \Sigma x_{3i} \left(Y_i - \hat{\beta}_1 - \hat{\beta}_2 x_{2i} - \hat{\beta}_3 x_{3i}\right) = 0$$

or, $\Sigma x_{3i} Y_i = \hat{\beta}_1 \Sigma x_{3i} + \hat{\beta}_2 \Sigma x_{2i} x_{3i} + \hat{\beta}_3 \Sigma x_{3i}^2$

or, $\Sigma x_{3i} Y_i = \left(\bar{Y} - \hat{\beta}_2 \bar{x}_2 - \hat{\beta}_3 \bar{x}_3\right) \Sigma x_{3i}$

$$+ \hat{\beta}_2 \left(\Sigma x_{2i} x_{3i}\right) + \hat{\beta}_3 \Sigma x_{3i}^2$$

or, $\underline{\Sigma x_{3} Y_i - \bar{Y} \Sigma x_{3i}} = \hat{\beta}_2 \left(\underline{\Sigma x_{2i} x_{3i} - \bar{x}_2 \Sigma x_{3i}}\right)$

$$+ \hat{\beta}_3 \left(\underline{\Sigma x_{3i}^2 - \bar{x}_3 \Sigma x_{3i}}\right)$$

or, $\Sigma x_3 y = \hat{\beta}_2 \Sigma x_2 x_3 + \hat{\beta}_3 \Sigma x_3^2 \longrightarrow (2)$

① − ②

$$\sum x_2 y = \hat{\beta}_2 \sum x_2^2 + \hat{\beta}_3 \sum x_2 x_3 \Rightarrow \hat{\beta}_2 = \frac{\sum x_2 y - \hat{\beta}_3 \sum x_2 x_3}{\sum x_2^2}$$

$$\sum x_3 y = \hat{\beta}_2 \sum x_2 x_3 + \hat{\beta}_3 \sum x_3^2$$

$$\sum x_3 y = \left[\left(\frac{\sum x_2 y - \hat{\beta}_3 \sum x_2 x_3}{\sum x_2^2}\right) \sum x_2 x_3 + \hat{\beta}_3 \sum x_3^2\right]$$

$$\sum x_3 y = \frac{\sum x_2 y \cdot \sum x_2 x_3 - \hat{\beta}_3 (\sum x_2 x_3)^2 + \hat{\beta}_3 \sum x_3^2 \sum x_2^2}{\sum x_2^2}$$

$$\sum x_3 y \cdot \sum x_2^2 = \sum x_2 y \cdot \sum x_2 x_3 - \hat{\beta}_3 (\sum x_2 x_3)^2 + \hat{\beta}_3 \sum x_3^2 \sum x_2^2$$

$$\sum x_3 y \cdot \sum x_2^2 - \sum x_2 y \sum x_2 x_3 = \hat{\beta}_3 \left[\sum x_3^2 \sum x_2^2 - (\sum x_2 x_3)^2\right]$$

$$\boxed{\hat{\beta}_3 = \frac{\sum x_3 y \cdot \sum x_2^2 - \sum x_2 x_3 \sum x_2 y}{\sum x_2^2 \sum x_3^2 - (\sum x_2 x_3)^2}}$$

$$\therefore \hat{\beta}_2 = \frac{\sum x_2 y - \hat{\beta}_3 \sum x_2 x_3}{\sum x_2^2}$$

$$\hat{\beta} = \sum x_2 y \quad \left[\sum x_3 y \cdot \sum x_2^2 - \sum x_2 x_3 \cdot \sum x_2 y\right] \sum x_2 x_3$$

$$\hat{\beta}_2 = \frac{\Sigma x_2 y}{\Sigma x_2^2} - \left[\frac{\Sigma x_3 y \cdot \Sigma x_2^2 - \Sigma x_2 x_3 \cdot \Sigma x_2 y}{\Sigma x_2^2 \Sigma x_3^2 - (\Sigma x_2 x_3)^2}\right] \frac{\Sigma x_2 x_3}{\Sigma x_2^2}$$

$$= \frac{1}{\Sigma x_2^2}\left[\frac{\Sigma x_2 y\left[\Sigma x_2^2 \Sigma x_3^2 - (\Sigma x_2 x_3)^2\right] - \Sigma x_3 y \cdot \Sigma x_2 x_3 \Sigma x_2^2 + (\Sigma x_2 x_3)^2 \Sigma x_2 y}{\Sigma x_2^2 \Sigma x_3^2 - (\Sigma x_2 x_3)^2}\right]$$

$$= \frac{1}{\Sigma x_2^2}\left[\frac{\Sigma x_2 y \, \Sigma x_2^2 \Sigma x_3^2 - \Sigma x_3 y \cdot \Sigma x_2 x_3 \Sigma x_2^2}{\Sigma x_2^2 \Sigma x_3^2 - (\Sigma x_2 x_3)^2}\right]$$

$$= \boxed{\frac{\Sigma x_2 y \, \Sigma x_3^2 - \Sigma x_3 y \cdot \Sigma x_2 x_3}{\Sigma x_2^2 \Sigma x_3^2 - (\Sigma x_2 x_3)^2} = \hat{\beta}_2}$$

Matrix Approach

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} \cdots + \beta_k X_{ki} + u_i$$

$$(i = 1, \ldots n)$$

$$\begin{cases} Y_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{21} + \beta_3 X_{31} + \cdots + \beta_k X_{k1} + U_1 \\ Y_2 = \beta_0 + \beta_1 X_{12} + \beta_2 X_{22} + \beta_3 X_{32} + \cdots + \beta_k X_{k2} + U_2 \\ Y_3 = \\ \vdots \\ Y_n = \beta_0 + \beta_1 X_{1n} + \beta_2 X_{2n} + \cdots + \beta_k X_{kn} + U_n \end{cases}$$

$$y_n = \beta_0 + \beta_1 x_{1n} + \beta_2 x_{2n} + \cdots + \beta_k x_{kn} + u_n$$

$$Y = X\beta + U$$

where $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times 1}$ $\quad X = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{k1} \\ 1 & X_{12} & X_{22} & & \vdots \\ 1 & X_{13} & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 1 & X_{1n} & X_{2n} & & X_{kn} \end{bmatrix}_{n\times(k+1)}$

β' $1\times(k+1)$

X' $(k+1)\times n$

$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}_{(k+1)\times 1}$ and $U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}_{n\times 1}$

$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$ $\quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$

$$e = Y - \hat{Y}$$

$$e = Y - X\hat{\beta}$$

$$\boxed{e = Y - X\hat{\beta}}$$

Minimise $\sum e_i^2 = e_1^2 + e_2^2 + \cdots + e_n^2$

$$\sum e_i^2 = [e_1 \; e_2 \cdots e_n] \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\sum e_i^2 = e'e$$

$$= (Y - X\hat{\beta})'(Y - X\hat{\beta})$$

$$= (Y - X\hat{\beta})'(Y - X\beta)$$

$$\Sigma e_i^2 = Y'Y - \hat{\beta}'X'Y - Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}$$

$$\Sigma e_i^2 = Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$e'e$

$\beta' X' \to 1 \times n$ $\quad Y \to n \times 1$

$\therefore \hat{\beta}'X'Y \to$ scalar value (1x1) matrix.

$= Y'X\hat{\beta}$ (same)

$$\frac{\partial \Sigma e_i^2}{\partial \hat{\beta}} = \frac{\partial (e'e)}{\partial \hat{\beta}} = -2X'Y + 2X'X\hat{\beta} = 0$$

$$X'X\hat{\beta} = X'Y$$

$$\boxed{\hat{\beta} = (X'X)^{-1}X'Y}$$

CLRM. $\quad Y_i = \alpha + \beta X_i + U_i$

$\Sigma(Y-\bar{Y})$ TSS $= \Sigma(\hat{Y}-\bar{Y})$ ESS $+ \Sigma e_i^2$ RSS

Random

$$\boxed{\hat{Y} = \hat{\alpha} + \hat{\beta} X}$$

$$(\hat{\text{cons}}) = \hat{\alpha} + \hat{\beta}(\text{inc}).$$

$R^2 = 0.70$

70% of variation in Y is explained by variation in X.

(cons) $\hat{\alpha}$ | in Y is explained by variation in x.

interceptterm ⇒ when income = 0
consum = $\hat{\alpha}$

$\hat{\beta}$ ⇒ slope of the consumption line

$$\Rightarrow \frac{\partial(\hat{cons})}{\partial(\text{income})} \left(\text{or } \frac{d\hat{y}}{\partial x} \right) =$$ For one unit increase in income consumpt exp increases by β units

$$\boxed{\Sigma(Y_i - \bar{Y})^2 = \hat{\beta}^2 \Sigma(X_i - \bar{X})^2 + \Sigma e_i^2}$$

TSS = ESS + RSS

$$\boxed{ESS = TSS - RSS}$$

$R^2 \rightarrow$ Coefficient of Determination.

$$\therefore R^2 = \frac{\text{Variations explained}}{\text{Variation required to be explained}} = \frac{ESS}{TSS}$$

$$\therefore R^2 = \frac{ESS}{TSS} = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$$

$$\boxed{RSS = TSS(1 - R^2).}$$

Adjusted R^2 is $\bar{R}^2$

↳ coeff of Determination.

↳ coeff of Determination.

(Both side divided by degree of freedom).

$(n-K)$ ↳ RSS

$(n-1) \to$ TSS

$$\frac{RSS}{(n-K)} = \frac{TSS}{(n-1)}(1-\bar{R}^2)$$

$$\left(\frac{n-1}{n-K}\right)\frac{RSS}{TSS} = 1-\bar{R}^2$$

$$\bar{R}^2 = 1 - \left(\frac{n-1}{n-K}\right)\frac{RSS}{TSS}$$

$$\bar{R}^2 = 1 - \frac{RSS/(n-K)}{TSS/n-1}$$