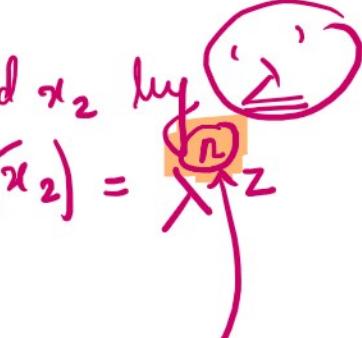


# Degree of Homogeneity of a function.

$$z = f(\underline{x}_1, \underline{x}_2)$$

Say let us multiply  $x_1$  and  $x_2$  by proportion, the  $f(\lambda \underline{x}_1, \lambda \underline{x}_2) = \underline{\lambda}^n z$



Ex.1: Let  $Q = 2L + 3K$  is a production function.

Comment on the degree of homogeneity and returns to scale.

Soln.: Let us increase  $L$  and  $K$  by ' $\lambda$ ' proportion, then we get,

$$\begin{aligned} & 2(\lambda L) + 3(\lambda K) \\ &= \lambda [2L + 3K] \\ &= \lambda^2 Q \quad [\because Q = 2L + 3K] \end{aligned}$$

∴ The function is homogeneous of degree 1 and follows constant returns to scale.

Ex.2:  $Q = A \cdot L^\alpha K^\beta$  is our production function.  
Do comment on degree of homogeneity & Returns to scale.

A -  $A L^\alpha K^\beta \longrightarrow ①$

n = degree of homogeneity.  
Classification of returns to scale.

①  $n=1$  :  $f(\lambda x_1, \lambda x_2) = \lambda z$   
[CRS  $\rightarrow$  Constant Returns to scale]

②  $n>1$  :  $f(\lambda x_1, \lambda x_2) < \lambda z$   
[IRS  $\rightarrow$  Increasing Returns to scale]

③  $n<1$  :  $f(\lambda x_1, \lambda x_2) > \lambda z$   
[DRS: Decreasing Returns to scale]

Q =  $A L^\alpha K^\beta \longrightarrow ①$

Suppose 'L' and 'K' is increased by  $\lambda$  proportion, then,

$$\begin{aligned} &= A (\lambda L)^\alpha (\lambda K)^\beta \\ &= A \lambda^\alpha L^\alpha \lambda^\beta K^\beta \\ &= \lambda^{\alpha+\beta} (A L^\alpha K^\beta) \end{aligned}$$

From ①,  $\lambda^{\alpha+\beta} \cdot Q$

$\therefore$  Degree of homogeneity is  $\alpha+\beta$ .

(i) If  $\alpha+\beta > 1 \rightarrow IRS$

(ii)  $\alpha+\beta < 1 \rightarrow DRS$

(iii)  $\alpha+\beta = 1 \rightarrow CRS$

## # Euler's theorem:

$z = f(x_1, x_2)$  with degree of homogeneity 'n'.

Then Euler's theorem states that,

$$f_1 \cdot x_1 + f_2 \cdot x_2 = n z$$

Ex3:  $Q = A L^\alpha K^\beta$ . Verify Euler's theorem.  
 From ex2  $\Rightarrow$  degree of homogeneity is  $(\alpha+\beta)$ .

from ex 2  $\Rightarrow$  degree of homogeneity  $\alpha + \beta$   
 To prove Euler's theorem we need  

$$\frac{\partial Q}{\partial L} \cdot L + \frac{\partial Q}{\partial K} \cdot K = (\alpha + \beta)Q$$

$$\begin{aligned}
 \text{LHS: } & \frac{\partial Q}{\partial L} \cdot L + \frac{\partial Q}{\partial K} \cdot K \\
 &= A \cancel{X} L^{\alpha-1} K^\beta \cdot \cancel{L} + A L^\alpha \cancel{X} K^{\beta-1} \cdot \cancel{K} \\
 &= \cancel{A X L}^{\alpha} \cancel{K}^{\beta} + \cancel{A B L}^{\alpha} \cancel{K}^{\beta} \\
 &= \underbrace{A L^\alpha K^\beta}_{Q} (\alpha + \beta) \\
 &= Q (\alpha + \beta) \\
 &= \text{RHS}
 \end{aligned}$$

verified

$$\text{Ex: } Q = 75 \left[ \underbrace{0.3 K^{-0.4}}_{\sim} + \underbrace{0.7 L^{-0.4}}_{\sim} \right]^{-2.5} \cdot \text{Verify Euler's theorem.}$$

Solution:

$$\begin{aligned}
 & \text{increase 'K' and 'L' by '}\lambda\text{' proportion,} \\
 & \text{we get, } 75 \left[ 0.3 (\lambda K)^{-0.4} + 0.7 (\lambda L)^{-0.4} \right]^{-2.5} \\
 &= 75 \left[ \underbrace{0.3 \lambda^{-0.4}}_{\sim} \cancel{K}^{-0.4} + \underbrace{0.7 \lambda^{-0.4}}_{\sim} \cancel{L}^{-0.4} \right]^{-2.5} \\
 &= \cancel{75} \lambda^{-0.4 (-2.5)} \left[ 0.3 \cancel{K}^{-0.4} + 0.7 \cancel{L}^{-0.4} \right]
 \end{aligned}$$

$$= 75 \lambda^{-0.4(-2.5)} [0.3K^{-0.4} + 0.7L^{-0.4}]$$

$$= \lambda \cdot Q$$

$\hookrightarrow$  degree of homogeneity is 1.

Euler's theorem states that:

$$\frac{\partial Q}{\partial L} \cdot L + \frac{\partial Q}{\partial K} \cdot K = 1 \cdot Q$$

LHS:  $\frac{\partial Q}{\partial L} \cdot L + \frac{\partial Q}{\partial K} \cdot K$

$$= 75 \times (-2.5) [0.3K^{-0.4} + 0.7L^{-0.4}]^{-1.4} \cdot L$$

$$+ 75 \times (-2.5) [0.3K^{-0.4} + 0.7L^{-0.4}]^{-1.4} \cdot K$$

$$= 75(-2.5) [0.3K^{-0.4} + 0.7L^{-0.4}]^{-3.5} \left[ -0.28L^{-0.4} - 0.12K^{-0.4} \right]^{-2.5}$$

$$= 75(-2.5) \left[ 0.3K^{-0.4} + 0.7L^{-0.4} \right]^{-2.5}$$

$$= 75 [0.3K^{-0.4} + 0.7L^{-0.4}]^{-2.5}$$

$$= 75 \left[ 0.3 K^{-0.4} + 0.7 L^{0.4} \right]$$

" = RHS      Verified

# Maximisation / Minimisation for  
2-variable functions.

i.e., 
$$Z = f(x_1, x_2)$$

Maximisation.

F.O.C:  $f_1 = 0, f_2 = 0$

S.O.C:  $\Rightarrow f_{11} < 0, f_{22} < 0$

$$\Rightarrow f_{11} \cdot f_{22} > f_{12}^2$$

Minimisation.

$$f_1 = 0, f_2 = 0$$

$$\Rightarrow f_{11} > 0, f_{22} > 0$$

$$\Rightarrow f_{11} \cdot f_{22} > f_{12}^2$$

Ex: find the stationary value & test the nature of  $x$  and  $y$  for the following fns:

$$Z = 3x^2 + 6xy + 7y^2$$

Soln

$$\text{F.O.L} \quad f_x = 6x + 6y = 0 \quad \text{--- (1)}$$

$$f_y = 6x + 14y = 0 \quad \text{--- (2)}$$

Solving (1) and (2),  $\begin{array}{r} 6x + 6y = 0 \\ 6x + 14y = 0 \\ \hline -8y = 0 \\ y = 0 \end{array}$

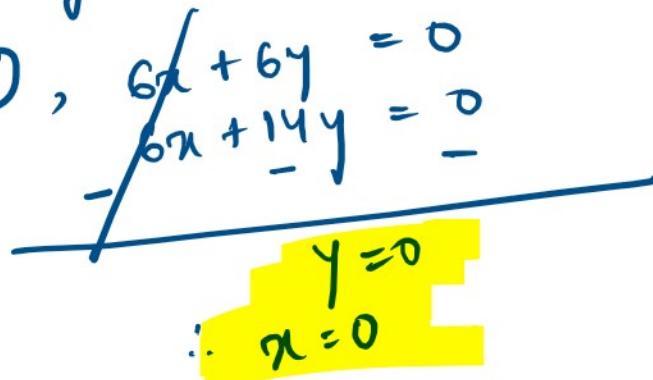
S.O.C

$$f_{xx} = 6 > 0, \quad f_{yy} = 14 > 0$$

$$f_{xy} = 6$$

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 14 \times 6 - 6^2 = 84 > 0$$

$\therefore$  S.O.C for minimisation is satisfied.  
 $\therefore$  at  $x=0, y=0 \rightarrow z$  is minimised. (Ans).



Ex:

A bi-product firm produces two commodities  $x$  and  $y$ , prices of which are 12 and 18 respectively. The joint cost fn is  $C = 2x^2 + xy + 2y^2$ . Obtain the optimal level of its products which maximise profit.

Solution

$$\text{Due to the ques, } TR, \text{ Rev, } TR = 12x + 18y \quad \text{--- (1)}$$

$$\text{and, } TC, C = 2x^2 + xy + 2y^2 \quad \text{--- (2)}$$

$$\therefore \text{Profit function, } \Pi = TR - TC$$

$$\Pi = (12x + 18y) - (2x^2 + xy + 2y^2)$$

∴  $\nabla \pi = 0$

$$\pi = (12x + 18y) - (2x^2 + xy + 2y^2)$$

$$\pi = 12x + 18y - 2x^2 - xy - 2y^2$$

$$\begin{aligned} \text{Now, } \pi_x &= 12 - 4x - y = 0 & - ① \times 4 \Rightarrow 16x + 4y = 48 \\ \pi_y &= 18 - x - 4y = 0 & - ② \Rightarrow x + 4y = 18 \\ && \hline & 15x = 30 \end{aligned}$$

S.O.C

$$\pi_{xx} = -4 < 0 \quad \pi_{yy} = -4 < 0$$

$$\pi_{xy} = -1$$

$$\therefore \begin{cases} x = 2 \text{ units} \\ y = 4 \text{ units} \end{cases}$$

$$\text{Now, } \pi_{xx} \cdot \pi_{yy} = 16 > (-1)^2$$

i.e.,  $\pi_{xx}\pi_{yy} > \pi_{xy}^2$

∴ S.O.C is satisfied for maximisation.  
∴ At  $x = 2$  units and  $y = 4$  units  
firm can maximise its profit.