Q. Let $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ [μ, σ^2 are unknown]: Use the likelihood ratio test to test:

$H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$, $H_1: \mu \neq \mu_0,\ 0 < \sigma^2 < \infty$.

$\Omega = \{(\mu, \sigma^2) \mid \mu \in \mathbb{R},\ 0 < \sigma^2 < \infty\}$. (Full-parameter space)

$\Omega_0 = \{(\mu, \sigma^2) \mid \mu = \mu_0,\ 0 < \sigma^2 < \infty\}$. (Parameter space under H_0)

$\therefore$ For likelihood Ratio Test $\lambda = \dfrac{L^{max}(\Omega_0)}{L^{max}(\Omega)}$

Testing Rule: Reject H_0 at $\alpha\%$ L.O.S if $\lambda \leq k$.

We know, $\boxed{\hat{\mu}_{MLE} = \bar{x},\quad \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \mu)^2}$.

$\therefore X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$f_{\mu,\sigma^2}(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}$$

$$\therefore L(\mu, \sigma^2) = \prod_{i=1}^{n} f_{\mu,\sigma^2}(x_i) = \boxed{\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i - \mu)^2}}$$

$$\boxed{L^{max}(\Omega)} = L(\hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE})$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i - \hat{\mu})^2}$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\hat{\sigma}^2}\boxed{\sum(x_i - \bar{x})^2}}$$

$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \mu)^2 = s^2$

$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$

$n\hat{\sigma}^2 = \sum(x_i - \bar{x})^2$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\cancel{\hat{\sigma}^2}}\cdot n\cancel{\hat{\sigma}^2}}$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-n/2} = \frac{1}{(\hat{\sigma}^2 2\pi)^{n/2}} e^{-n/2} = \frac{1}{(2\pi s^2)^{n/2}} e^{-n/2}$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-n/2} = \frac{1}{(\hat{\sigma}^2 \cdot 2\pi)^{n/2}} e^{-n/2} = \frac{1}{(2\pi s^2)^{n/2}} e^{-n/2}$$

$$L^{max}(\Omega_0) = L(\mu_0, \hat{\sigma}^2_{MLE})$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\mu_0)^2}$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\cancel{\hat{\sigma}^2}} \cdot n \cancel{\hat{\sigma}^2}}$$

$$= \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-n/2} = \frac{1}{(\hat{\sigma}^2 2\pi)^{n/2}} e^{-n/2} = \frac{1}{(s_0^2 2\pi)^{n/2}} e^{-n/2}$$

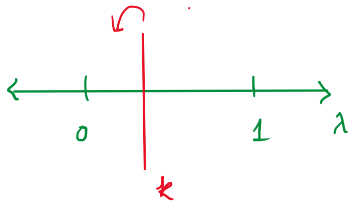
$$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu)^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i-\mu_0)^2 = s_0^2$$

$$n\hat{\sigma}^2 = \sum(x_i-\mu_0)^2$$

$$\therefore \lambda = \frac{L^{max}(\Omega_0)}{L^{max}(\Omega)} = \frac{\frac{1}{(2\pi s_0^2)^{n/2}} \cancel{e^{-n/2}}}{\frac{1}{(2\pi s^2)^{n/2}} \cancel{e^{-n/2}}} = \left(\frac{s^2}{s_0^2}\right)^{n/2}$$

Reject H_0 at $\alpha\%$ L.o.s if $\lambda \leq k$.



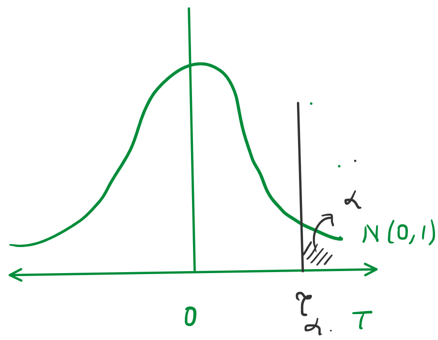
$$\alpha = P[\lambda \leq k \mid H_0]$$

$$\alpha = \int_0^k g(\lambda|H_0)\, d\lambda \quad [\text{where } g(\cdot) \text{ is the pdf of } \lambda]$$

Eg: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $[\sigma^2 = \text{known}]$

To test: $H_0: \mu = \mu_0$ vs $H_1: \mu > \mu_0$

$\therefore$ Test statistic $T = \dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$



How to calculate τ_α :-

(i) Know the sampling distn of T under H_0.

(ii) Since $T \sim N(0,1) \Rightarrow f(t) = \dfrac{1}{\sqrt{2\pi}} e^{-t^2/2}$, $-\infty < t < \infty$

$\dots N(0,1) \Rightarrow f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad -\infty < t < \infty.$

$\therefore \left\{ \alpha = \int_{\tau_\alpha}^{\infty} f(t)\,dt \right\} \Rightarrow$ solve this to get τ_α.

Result

Let $Y_1, Y_2, \cdots, Y_n$ have joint likelihood function $L(\Theta)$. Then, for large n, $-2\ln(\lambda)$ converges to a chi-squared distribution. The degrees of freedom of the limiting distribution is the difference between the number of free parameters specified by $\Theta \in \Omega_0$ and the number of free parameters specified by $\Theta \in \Omega$.

Confidence Intervals

Under Hypothesis Testing:

i) Testing for Mean [σ^2 is known].

$$T = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

ii) Testing for Mean [σ^2 is unknown]

$$T = \frac{\bar{X} - \mu}{s'/\sqrt{n}} \sim t_{(n-1)}$$

iii) Testing for Population Proportion

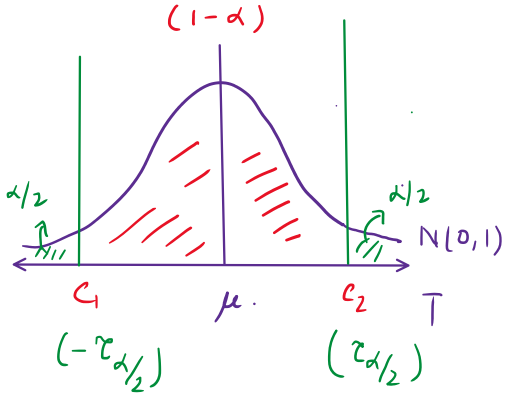
$$T = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \overset{a}{\sim} N(0,1)$$

Q. Construct $100(1-\alpha)\%$ confidence interval for the unknown popln parameter in each of the above cases.

(i) CI for Mean [σ^2 is k...] $(1-\alpha)$

... Parameter in each of the above cases.

(i) CI for Mean [σ^2 is known]



$$T = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Construct an interval $\mu \in [c_1, c_2]$ s.t Prob of the true value of μ lying outside this interval is α. This will give us $100(1-\alpha)\%$ CI of μ.

For constructing $100(1-\alpha)\%$ C.I of μ:-

$$P\left[-\tau_{\alpha/2} \leq T \leq \tau_{\alpha/2}\right] = (1-\alpha).$$

$$\Rightarrow P\left[-\tau_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq \tau_{\alpha/2}\right] = (1-\alpha)$$

$$\Rightarrow P\left[-\frac{\sigma}{\sqrt{n}}\cdot\tau_{\alpha/2} \leq (\bar{X}-\mu) \leq \frac{\sigma}{\sqrt{n}}\cdot\tau_{\alpha/2}\right] = (1-\alpha)$$

$$\Rightarrow P\left[\bar{X}-\frac{\sigma}{\sqrt{n}}\cdot\tau_{\alpha/2} \leq \mu \leq \bar{X}+\frac{\sigma}{\sqrt{n}}\tau_{\alpha/2}\right] = (1-\alpha).$$

$\therefore$ $100(1-\alpha)\%$ C.I of $\mu \in \left[\bar{X}-\frac{\sigma}{\sqrt{n}}\tau_{\alpha/2},\ \bar{X}+\frac{\sigma}{\sqrt{n}}\tau_{\alpha/2}\right]$.

(ii) Compute $100(1-\alpha)\%$ C.I of μ where σ^2 is unknown

$$\therefore\ T = \frac{\bar{X}-\mu}{s'/\sqrt{n}} \sim t_{(n-1)}$$

$$\therefore\ P\left[-t_{\alpha/2;(n-1)} \leq \frac{\bar{X}-\mu}{s'/\sqrt{n}} \leq t_{\alpha/2;(n-1)}\right] = (1-\alpha)$$ [Solve for μ]

(iii) Compute $100(1-\alpha)\%$ C.I of p:-

$$\therefore\ T = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \overset{a}{\sim} N(0,1)$$

$P[\_\_ \leq p \leq \_\_] = (1-\alpha)$

$$\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1) \qquad P[\text{—} \le p \le \text{—}] = (1-\alpha)$$

$$\therefore P\left[-z_{\alpha/2} \le \frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \le z_{\alpha/2}\right] = (1-\alpha)$$

$$\Rightarrow P\left[-\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2} \le \hat{p}-p \le \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2}\right] = (1-\alpha).$$

$$\Rightarrow P\left[\hat{p}-\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2} \le p \le \hat{p}+\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2}\right] = (1-\alpha)$$

$$\therefore 100(1-\alpha)\%\ \text{C.I of } p \in \left[\hat{p}-\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2},\ \hat{p}+\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\, z_{\alpha/2}\right]$$

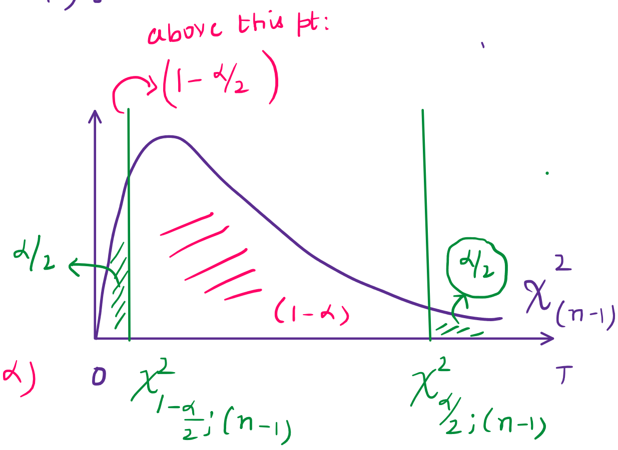
iv) Construct the $100(1-\alpha)\%$ C.I for population variance (σ^2) when μ is unknown.

Recall: $$\frac{\sum(x_i-\bar{x})^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$\therefore$ Sample variance $$s'^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$$

$$\Rightarrow \sum(x_i-\bar{x})^2 = (n-1)s'^2$$

$$\therefore T = \frac{(n-1)s'^2}{\sigma^2} \sim \chi^2_{(n-1)}$$



Construct $100(1-\alpha)\%$ C.I of σ^2:-

$$\therefore P\left[\chi^2_{1-\alpha/2;(n-1)} \le \frac{(n-1)s'^2}{\sigma^2} \le \chi^2_{\alpha/2;(n-1)}\right] = (1-\alpha)$$

$$\Rightarrow P\left[\frac{1}{\chi^2_{1-\alpha/2;(n-1)}} \ge \frac{\sigma^2}{(n-1)s'^2} \ge \frac{1}{\chi^2_{\alpha/2;(n-1)}}\right] = (1-\alpha)$$

$$\Rightarrow P\left[\frac{(n-1)s'^2}{\chi^2_{1-\alpha/2;(n-1)}} \geq \sigma^2 \geq \frac{(n-1)s'^2}{\chi^2_{\alpha/2;(n-1)}}\right] = (1-\alpha)$$

$$\Rightarrow P\left[\frac{(n-1)s'^2}{\chi^2_{\alpha/2;(n-1)}} \leq \sigma^2 \leq \frac{(n-1)s'^2}{\chi^2_{1-\alpha/2;(n-1)}}\right] = (1-\alpha)$$

$\therefore$ 100$(1-\alpha)$% C.I of $\sigma^2 \in \left[\frac{(n-1)s'^2}{\chi^2_{\alpha/2;(n-1)}}, \frac{(n-1)s'^2}{\chi^2_{1-\alpha/2;(n-1)}}\right]$

e.g 6:

Consider the large hospital that wants to estimate the average length of stay of its patients, μ. The hospital length of stay follows a normal distribution. The hospital randomly samples $n = 20$ of its patients and finds that the sample mean length of stay is $\bar{x} = 4.5$ days. Also, suppose it is known that the standard deviation of the length of stay for all hospital patients is $\boxed{\sigma = 4 \text{ days}}$. Construct 90% confidence interval for the target parameter, μ.

$n = 20$, $\bar{x} = 4.5$, $\sigma = 4$, $\alpha = 0.1$.

$$P\left[-\tau_{0.05} \leq \frac{4.5-\mu}{4/\sqrt{20}} \leq \tau_{0.05}\right] = 0.9. \Rightarrow \text{solve for } \mu.$$

e.g 8:

An experimenter wanted to check the variability of measurements obtained by using equipment designed to measure the volume of an audio source. Three independent measurements recorded by this equipment for the same sound were 4.1, 5.2, and 10.2. Estimate σ^2 with confidence coefficient .90.

$$T = \frac{(n-1)s'^2}{\sigma^2} = \frac{\sum(x_i-\bar{x})^2}{\sigma^2}.$$

$x_1 = 4.1$

$x_2 = 5.2$

$x_3 = 10.2$.

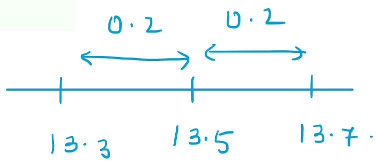
$$= \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + (x_3-\bar{x})^2}{\sigma^2}. \qquad \bar{x} = \frac{x_1+x_2+x_3}{3}.$$

$$P\left[\chi^2_{0.95;(2)} \leq \frac{\quad}{\sigma^2} \leq \chi^2_{0.05;(2)}\right] = 0.9 \ldots \text{[solve for } \sigma^2\text{]}$$

$$P\left[\chi^{2}_{0.95;(2)} \le \frac{}{\sigma^2} \le \chi^{2}_{0.05;(2)}\right] = 0.9 \quad \ldots \text{ [solve for } \sigma^2 \text{]}$$

e.g 7:

Suppose the manufacturer of official NFL footballs uses a machine to inflate the new balls to a pressure of 13.5 pounds. When the machine is properly calibrated, the mean inflation pressure is 13.5 pounds, but uncontrollable factors cause the pressures of individual footballs to vary randomly from about 13.3 to 13.7 pounds. For quality control purposes, the manufacturer wishes to estimate the mean inflation pressure to within 0.025 pound of its true value with a 99% confidence interval. What sample size should be specified for the experiment?



$\mu = 13.5$, $\mu \in [13.3, 13.7]$

$$P\left[\bar{X} - 0.025 \le \mu \le \bar{X} + 0.025\right] = 0.99 \quad \ldots \text{ solve for 'n'.}$$

$1 - \alpha = 0.99 \Rightarrow \alpha = 0.01$.

Recall: $$P\left[\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right] = (1-\alpha)$$

$$z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 0.025 \quad , \quad \alpha = 0.01$$

$$\boxed{z_{0.005} \frac{\sigma}{\sqrt{n}} = 0.025} \Rightarrow \text{solve for 'n'.}$$

$$\frac{z_{0.05}\,\sigma}{0.025} = \sqrt{n} \Rightarrow n = \frac{\sigma^2 z^2_{0.05}}{(0.025)^2}$$

$$P\left[\bar{X} - c \le \mu \le \bar{X} + c\right] = (1-\alpha).$$

Margin of Error (ME)