Infinite series

Suppose we have seq $\{u_n\}$. Series constructed from the given seq : $(u_1 + u_2 + u_3 + \cdots\cdots) = \sum_{n=1}^{\infty} u_n$.

(*) p-series $:= \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$ $> 1 \Rightarrow$ convergent, $\leq 1 \Rightarrow$ divergent.

Convergence of series $\Rightarrow$ Existence of finite sum, i.e. $\sum_{n=1}^{\infty} u_n = S$

Tests for convergence of series :-

(1) Comparision test : Consider $\sum u_n$ and $\sum v_n$ be 2 series of positive terms.

as $n \to \infty$ $u_n < v_n \Rightarrow \sum u_n < \sum v_n$

$\Rightarrow$ If $\left(\lim_{n \to \infty} \frac{u_n}{v_n}\right) \to 0$. [If $\sum v_n$ is convergent, then $\sum u_n$ is also convergent].

$\Rightarrow$ If $\left(\lim_{n \to \infty} \frac{u_n}{v_n}\right) \to \infty$. [If $\sum v_n$ is divergent, then $\sum u_n$ is also divergent].

Q. $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{n + \sqrt{n}}{2n^3 + 4n - 1}$. Test for convergence.

$$u_n = \frac{n + \sqrt{n}}{2n^3 + 4n - 1}$$

$$v_n = \frac{n}{n^3} = \frac{1}{n^2} \quad \left[\frac{\text{Highest pow of num}}{\text{Highest pow of deno}} \right]$$

$\sum v_n = \sum \frac{1}{n^2} \Rightarrow$ p-series with $p = 2$ (convergent)

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{(n + \sqrt{n})\, n^2}{2n^3 + 4n - 1} = \frac{1}{2} < 1.$$

$\therefore \left(\lim_{n \to \infty} \frac{u_n}{v_n}\right)$ is finite & $\sum v_n$ is convergent $\Rightarrow$

$\sum u_n$ is convergent.

Q. $\sum u_n = \frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots$

$\quad\quad (1) \quad\quad (2) \quad\quad (3) \quad + \cdots$

$$u_n = \frac{1+2+\cdots+(n+1)}{(n+1)^3} = \frac{(n+1)(n+2)/2}{(n+1)^3} = \frac{(n+2)}{2(n+1)^2}$$

Consider $v_n = \frac{n}{n^2} = \frac{1}{n}$, $\sum v_n = \sum \frac{1}{n^1}$ -- p-series with $p = 1 \Rightarrow$ Divergent

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(n+2)n}{2(n+1)^2} = \frac{1}{2} < 1.$$

As $n \to \infty$. $u_n < v_n \Rightarrow \sum u_n < \sum v_n$

$\sum v_n$ divergent

(2) D-Alembert's Ratio Test:

Consider series $\sum u_n$ of positive terms.

Evaluate: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$
- $l < 1 \Rightarrow$ convergent
- $l > 1 \Rightarrow$ divergent
- $l = 1 \Rightarrow$ inconclusive (test fails)

(3) Raabe's Test:

Consider series $\sum u_n$ of positive terms.

Evaluate: $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$
- $l > 1 \Rightarrow$ convergent
- $l < 1 \Rightarrow$ divergent
- $l = 1 \Rightarrow$ inconclusive

(4) Cauchy's Root Test:

Consider series $\sum u_n$ of positive terms.

Evaluate: $\lim (u_n)^{1/n}$
- $l < 1 \Rightarrow$ convergent

$\dots\dots \sum u_n$ of positive terms.

Evaluate: $\lim_{n\to\infty} (u_n)^{1/n} = l$

$\to l < 1 \Rightarrow$ convergent

$\to l > 1 \Rightarrow$ divergent

$\to l = 1 \Rightarrow$ inconclusive.

Q. $$\sum u_n = \sum \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-n}$$

$$u_n = \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-n}$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{ \left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right) \right\}^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]}$$

$$= \frac{1}{e-1} < 1 \quad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e\right]$$

$\therefore$ By Cauchy Root Test: $\sum u_n$ is convergent.

Q. $$\sum u_n = \frac{a}{b} + \frac{1+a}{1+b} + \frac{(1+a)(2+a)}{(1+b)(2+b)} + \dots\dots$$

$$u_n = \frac{(1+a)(2+a)\cdots(n-1+a)}{(1+b)(2+b)\cdots(n-1+b)}$$

$$u_{n+1} = \frac{(1+a)(2+a)\cdots(n-1+a)(n+a)}{(1+b)(2+b)\cdots(n-1+b)(n+b)}$$

$$u_{n+1} = \frac{\quad}{(1+b)(2+b)\cdots(n-1+b)(n+b)}$$

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{n+b}{n+a} - 1\right] = \lim_{n\to\infty} n\left[\frac{b-a}{n+a}\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{1+\frac{a}{n}} = b-a.$$

Raabe's Test

If $b-a > 1 \Rightarrow b > (a+1) \Rightarrow$ convergent.

$b-a < 1 \Rightarrow b < (a+1) \Rightarrow$ divergent.

HW. Perform D-Alembert's Ratio Test on u_n.