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(1) Let S is non-empty convex subset of R^n and $f : S \to R$ be a function. Then:

(i) If 'f' is convex on S, Then f is quasi-convex on S.

<u>Proof</u>: Let f be convex.

Then for $x_1, x_2 \in S$ and $\lambda \in [0,1]$ we have,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

Let $\alpha = \text{Max}\{f(x_1), f(x_2)\}$

Then $f(x_1) \le \alpha$, $f(x_2) \le \alpha$

$$\therefore f(\lambda x_1 + (1-\lambda)x_2) \le \lambda\alpha + (1-\lambda)\alpha$$

ie $f(\lambda x_1 + (1-\lambda)x_2) \le \alpha$

ie $f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\}$

$\therefore$ f is quasi-convex on S.

(ii) If 'f' is concave, then 'f' is quasi-concave on S.

Proof Let 'f' be concave.

Then for $x_1, x_2 \in S$ and $\lambda \in [0,1]$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Let $\alpha \leq \min\{f(x_1), f(x_2)\}$

Then $\alpha \leq f(x_1)$ and $\alpha \leq f(x_2)$

$\therefore f(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$

or, $f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$

$\therefore$ f is quasiconcave on S.

③ Let S be a non empty convex subset of R^n and $f : S \to R$ be a function. Then:

(i) f is quasi-convex on S iff the lower set

$$S_\alpha = \{x \in S : f(x) \leq \alpha\} \text{ is convex for all } \alpha \in R.$$

(i) Let f be quasiconvex on S and $\alpha \in R$.

Let $x_1, x_2 \in S_\alpha = \{x \in S : f(x) \leq \alpha\}$ and $\lambda \in (0,1)$.

Since S is a convex set

$\therefore \{\lambda x_1 + (1-\lambda)x_2\} \in S$

By def of quasiconvex!

$f(\ldots)$ ... f is quasiconvex

Now if f is quasi convex, ← By def of quasiconvex!

$$\therefore f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\} \leq \alpha$$

$$\Rightarrow \quad \lambda x_1 + (1-\lambda)x_2 \in S_\alpha$$

$\therefore$ S_α is a convex set.

Conversely let S_α be convex set for each real number α (i.e $\alpha \in R$)

Then, $x_1, x_2 \in S$

let $\alpha = \max\{f(x_1), f(x_2)\}$

$\therefore$ $f(x_1) \leq \alpha$ and $f(x_2) \leq \alpha$

$\Rightarrow x_1, x_2 \in S_\alpha$

$\lambda x_1 + (1-\lambda)x_2 \in S_\alpha$

and $f(\lambda x_1 + (1-\lambda)x_2) \leq \alpha$

$f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\}$

$\therefore$ f is quasiconvex on S.

(ii) If 'f' is quasiconcave on S, iff upper set $S_\alpha = \{x \in S : f(x) \geq \alpha\}$ is convex set.

Let f be quasi concave on S.

Let $x_1, x_2 \in S^\alpha = \{x \in S : f(x) \geq \alpha\}$ and $\lambda \in (0,1)$

Then $x_1, x_2 \in S$, $f(x_1) \geq \alpha$ and $f(x_2) \geq \alpha$

Since S is convex $\therefore \lambda x_1 + (1-\lambda) x_2 \in S$

Now, f is quasiconcave on S

$$\therefore f(\lambda x_1 + (1-\lambda) x_2) \geq \min\{f(x_1), f(x_2)\} \geq \alpha$$

(By def of quasiconcave)

$$\therefore \lambda x_1 + (1-\lambda) x_2 \in S_\alpha$$

Hence S_α is a convex set.

Conversely, Let S^α be a convex set for each real number α.

Let $x_1, x_2 \in S$.

Let $\alpha = \min\{f(x_1), f(x_2)\}$

Then $f(x_1) \geq \alpha$ and $f(x_2) \geq \alpha$ $\}$ $x_1, x_2 \in S_\alpha$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S_\alpha$

then $\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$ and $\lambda \in (0,1]$

$$f(\lambda x_1 + (1-\lambda) x_2) \geq \alpha$$

$f(\lambda x_1 + (1-\lambda) x_2) \geq \alpha$

By def : $f(\lambda x_1 + (1-\lambda) x_2) \geq \text{Min}\{f(x_1) f(x_2)\} \geq \alpha$

$\therefore$ f is quasiconcave on S.

⑤ Let S be a non-empty convex subset of R^n and $f : S \to R$ be a function.

Then (i) f is quasiconvex on S iff

$$f(\bar{x}) \geq f(x) \Rightarrow f(\bar{x}) \geq f(\lambda \bar{x} + (1-\lambda) x)$$

for all $x, \bar{x} \in S$.

(ii) f is quasiconcave iff

$$f(\bar{x}) \leq f(x)$$

$$\Rightarrow f(\bar{x}) \leq f(\lambda \bar{x} + (1-\lambda) x) \text{ for all } x, \bar{x} \in S.$$

(i) Let f be a quasiconvex on S.

Let $f(\bar{x}) \geq f(x)$ for all $\bar{x}, x \in S$.

Since f is a quasiconvex on S

$\therefore$ S_α is a convex set

Let $\alpha = f(\bar{x})$

Then $x, \bar{x} \in S_\alpha$

$\Rightarrow \lambda\bar{x} + (1-\lambda)x \in S_\alpha$

$\Rightarrow f(\lambda\bar{x} + (1-\lambda)x) \le \alpha$

$\Rightarrow f(\lambda\bar{x} + (1-\lambda)x) \le f(\bar{x})$

ie, $f(\bar{x}) \ge f(\lambda\bar{x} + (1-\lambda)x) \ \forall \lambda \in [0,1]$.

Conversely, let $f(\bar{x}) \ge f(x)$

$\Rightarrow f(\bar{x}) \ge f(\lambda\bar{x} + (1-\lambda)x)$

We are to prove that f is quasiconvexous.

Let $x, \bar{x} \in S_\alpha$ and $f(\bar{x}) > f(x)$

Then $x, \bar{x} \in S$

$f(x) \le \alpha$

then - $f(\bar{x}) \le \alpha$

$\Rightarrow f(\bar{x}) \ge f(\lambda\bar{x} + (1-\lambda)x)$ (assumed above)

$\Rightarrow f(\lambda\bar{x} + (1-\lambda)x) \le f(\bar{x}) \le \alpha$

$\Rightarrow \lambda\bar{x} + (1-\lambda)x \in S_\alpha \quad \forall \lambda \in (0,1)$

$\Rightarrow S_\alpha$ is convex set

and f is quasi convex.