Elasticity of Substitution (σ): [of a Production function]

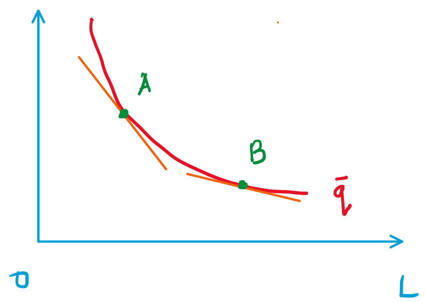
Given $q = f(L, K)$; $\underbrace{\frac{\partial q}{\partial L}}_{MP_L} > 0$, $\underbrace{\frac{\partial q}{\partial K}}_{MP_K} > 0$.

$$\sigma = \frac{\%\Delta (K/L)}{\%\Delta (MRTS)}.$$

where MRTS = Abs slope of the isoquant

Defn $MRTS = \dfrac{MP_L}{MP_K}$.

technologically feasible L/K rate of substitution



$$\sigma = \frac{d(K/L)/(K/L)}{d(MRTS)/(MRTS)}$$

$$\sigma = \frac{d[\ln(K/L)]}{d[\ln(MRTS)]}$$

$$\frac{dx}{x} = d\underbrace{[\ln x]}_{= \frac{1}{x}\cdot dx}$$

Q. Find the elasticity of substitution for $q = A L^{\alpha} K^{\beta}$

$$MP_L = \frac{\partial q}{\partial L} = A \cdot \alpha L^{\alpha - 1} K^{\beta}.$$

$$MP_K = \frac{\partial q}{\partial K} = A \beta L^{\alpha} K^{\beta - 1}$$

$$MRTS = \frac{MP_L}{MP_K} = \left(\frac{\alpha}{\beta}\right)\left(\frac{K}{L}\right)$$

$$\ln(MRTS) = \ln\left(\frac{\alpha}{\beta}\right) + \ln\left(\frac{K}{L}\right)$$

Diff: $d[\ln(MRTS)] = d\left[\ln\left(\frac{K}{L}\right)\right]$

$$1 = \frac{d\left[\ln\left(\frac{K}{L}\right)\right]}{d[\ln(MRTS)]} = \sigma \Rightarrow \boxed{\sigma = 1}$$

Q. Find the elasticity of substitution for:

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$$q = A\left[\delta L^{-\rho} + (1-\delta) K^{-\rho}\right]^{-1/\rho}, \quad \delta > 0.$$

$$MP_L = \frac{\partial q}{\partial L} = A\left(-\frac{1}{\rho}\right)[\cdot]^{-\frac{1}{\rho}-1}\delta(-\rho)L^{-\rho-1}$$

$$MP_K = \frac{\partial q}{\partial K} = A\left(-\frac{1}{\rho}\right)[\cdot]^{-\frac{1}{\rho}-1}(1-\delta)\cdot(-\rho)\cdot K^{-\rho-1}$$

$$MRTS = \frac{MP_L}{MP_K} = \left(\frac{\delta}{1-\delta}\right)\frac{L^{-\rho-1}}{K^{-\rho-1}} = \left(\frac{\delta}{1-\delta}\right)\left(\frac{K}{L}\right)^{(\rho+1)}$$

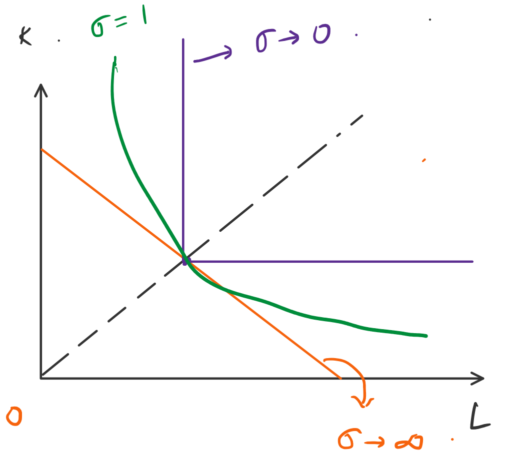
$$\ln(MRTS) = \ln\left(\frac{\delta}{1-\delta}\right) + (1+\rho)\cdot\ln\left(\frac{K}{L}\right)$$

Diff: $d[\ln(MRTS)] = (1+\rho)\cdot d\left[\ln\left(\frac{K}{L}\right)\right]$

$$\frac{1}{1+\rho} = \frac{d[\ln(K/L)]}{d[\ln(MRTS)]} = \sigma$$

$$\boxed{\sigma = \frac{1}{1+\rho}}$$

CES = Constant Elasticity of Substitution



(i) $\rho = -1$ [Perfect Substitutes].

we find $\sigma \to \infty$, i.e higher σ greater is the degree of substitution b/w K & L in the production process.

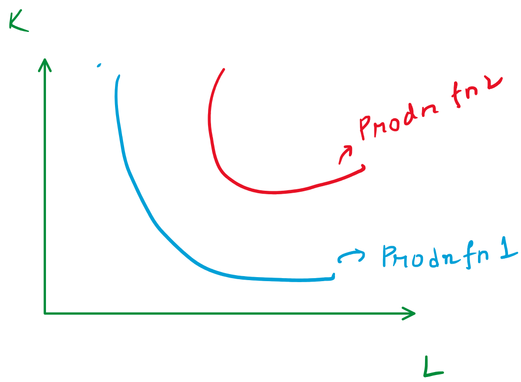
(ii) $\rho \to \infty$, $\sigma \to 0$ [Perfect complements].

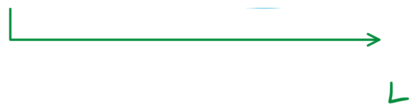
(iii) $\rho \to 0$, $\sigma \to 1$ [Cobb-Douglas]

Note: Suppose: $q = AL^{\alpha}K^{\beta}$, $\alpha, \beta > 0$.

$\alpha = \frac{MP_L \cdot L}{q}$ = share of labour in production

$\beta = \frac{MP_K \cdot K}{q}$ = share of capital in the production.





$\beta = \dfrac{MP_K \cdot K}{q}$ = share of capital in the production.

$\alpha + \beta$ = share of the factors of production in the production process.

$$\alpha + \beta = \frac{MP_L \cdot L}{q} + \frac{MP_K \cdot K}{q} = \frac{MP_L \cdot L + MP_K \cdot K}{q}.$$

$$\Rightarrow \quad MP_L \cdot L + MP_K \cdot K = \underbrace{(\alpha+\beta)}_{\text{degree of homogeneity of the prodn fn.}} \cdot q$$

$\alpha + \beta > 1 \Rightarrow IRS$

$\alpha + \beta = 1 \Rightarrow CRS$

$\alpha + \beta < 1 \Rightarrow DRS$.

$$\therefore \left.\begin{array}{lll} \text{(i) } IRS: & & MP_L \cdot L + MP_K \cdot K > q \\ \text{(ii) } CRS: & & MP_L \cdot L + MP_K \cdot K = q \\ \text{(iii) } DRS: & & MP_L \cdot L + MP_K \cdot K < q \end{array}\right\} \text{HW}$$