Q. Find the coeff of $x^{950}$ in the expansion of:

$$(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \cdots + x^{1000}.$$

(a) ${}^{1000}C_{950}$ (b) ${}^{1001}C_{950}$ (c) ${}^{1001}C_{949}$ (d) None.

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n \cdot x^n.$$

$(1+x)^{1000} \Rightarrow$ coeff of $x^{950} = {}^{1000}C_{950}$.

$\underbrace{x(1+x)^{999}}_{\text{Look for } x^{949}} \Rightarrow$ coeff of $x^{950} = {}^{999}C_{949}$.

Total coeff of $x^{950} = {}^{1000}C_{950} + {}^{999}C_{949} + {}^{998}C_{948} + \cdots + {}^{50}C_0$.

$$= {}^{1000}C_{950} + \cdots + {}^{52}C_2 + {}^{51}C_1 + {}^{50}C_0.$$

$$= {}^{1000}C_{950} + {}^{999}C_{949} + {}^{998}C_{948} + \cdots$$
$$+ \cdots + {}^{52}C_2 + {}^{51}C_1 + {}^{50}C_0.$$

$a = (1+x)^{1000}$, $r = \dfrac{x}{1+x}$

$$\Sigma S_n = \frac{(1+x)^{1000} \cdot \left[1 - \left(\frac{x}{1+x}\right)^{1001}\right]}{1 - \left(\frac{x}{1+x}\right)} = (1+x)^{1001} - x^{1001}.$$

Coeff of $x^{950}$ as: ${}^{1001}C_{950}$.

Q. $\displaystyle\sum_{r=0}^{m} {}^{n+r}C_r = ?$

(a) ${}^{n+m+1}C_{n+m}$ (c) ${}^{m+n+1}C_{n+1}$

(b) ${}^{n+m+1}C_n$ (d) None.

$$\underbrace{{}^nC_0}_{=1} + {}^{n+1}C_1 + {}^{n+2}C_2 + {}^{n+3}C_3 + \cdots + {}^{n+m}C_m.$$

$$\left({}^{n+1}C_0 + {}^{n+1}C_1\right) + {}^{n+2}C_2 + {}^{n+3}C_3 + \cdots + {}^{n+m}C_m .$$

Note: ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r .$

$$\left({}^{n+2}C_1 + {}^{n+2}C_2\right) + {}^{n+3}C_3 + \cdots + {}^{n+m}C_m .$$

$$\left({}^{n+3}C_2 + {}^{n+3}C_3\right) + \cdots + {}^{n+m}C_m .$$

$$\cdots\cdots\cdots$$

$$ {}^{n+m}C_{m-1} + {}^{n+m}C_m . = {}^{m+n+1}C_m$$

Note: ${}^nC_r = {}^nC_{n-r}$ .

$$= {}^{m+n+1}C_{\cancel{m}+n+1-\cancel{m}}$$

$$= {}^{m+n+1}C_{n+1} . \quad (c)$$

Q. $\displaystyle\sum_{i=0}^{n}\sum_{j=1}^{n} {}^nC_j \, {}^jC_i = ?$

(a) $3^n - 1$ (c) $3^n + 1$

(b) $2^n - 1$ (d) None .

$$\sum_{i=0}^{n}\left[{}^nC_1 \, {}^1C_i + {}^nC_2 \, {}^2C_i + {}^nC_3 \, {}^3C_i + \cdots + {}^nC_n \cdot {}^nC_i\right] .$$

$${}^nC_1 \underbrace{\sum_{i=0}^{n} {}^1C_i}_{\left({}^1C_0 + {}^1C_1\right) = 2^1} + {}^nC_2 \underbrace{\sum_{i=0}^{n} {}^2C_i}_{\left({}^2C_0 + {}^2C_1 + {}^2C_2\right) = 2^2} + {}^nC_3 \underbrace{\sum_{i=0}^{n} {}^3C_i}_{2^3} + \cdots + {}^nC_n \underbrace{\sum_{i=0}^{n} {}^nC_i}_{2^n}$$

$n=1$ $\quad$ $n=2$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n \cdot x^n$$

Put $x=1$ : $2^n = \left({}^nC_0 + {}^nC_1 + \cdots + {}^nC_n\right)$

Put $n=1$ :- $2^1 = {}^1C_0 + {}^1C_1$

Put $n=2$ :- $2^2 = {}^2C_0 + {}^2C_1 + {}^2C_2$

Put $x=2$

$$3^n = {}^nC_0 + \left({}^nC_1 \cdot 2^1 + {}^nC_2 \cdot 2^2 + \cdots + \cdots + {}^nC_n \cdot 2^n\right)$$

$$= \left({}^nC_1 2^1 + {}^nC_2 \cdot 2^2 + {}^nC_3\, 2^3 + \cdots + {}^nC_n \cdot 2^n + {}^nC_0\right) - {}^nC_0$$

$$= 3^n - {}^nC_0 = 3^n - 1 \quad (a)$$

Hypergeometric Sum :-

$$\sum_{i=0}^{m} {}^{p}C_i \cdot {}^{q}C_{m-i} = \boxed{p+q}C_m$$

Q. Find the value of : $\left\{ \sum_{i=0}^{100} {}^kC_i \, {}^{m-k}C_{100-i} \cdot \frac{k-i}{m-100} \right\} / {}^mC_{100}$

(a) $\frac{k}{m}$ (b) $\frac{m}{k}$ (c) $mk$ (d) None

$$\frac{1}{{}^mC_{100}} \sum_{i=0}^{100} {}^kC_i \; {}^{m-k}C_{100-i} \left(\frac{k-i}{m-100}\right)$$

$$\frac{1}{{}^mC_{100}\,(m-100)} \sum_{i=0}^{100} {}^kC_i \; {}^{m-k}C_{100-i} \; (k-i)$$

$$\frac{1}{{}^mC_{100}\,(m-100)} \sum_{i=0}^{100} \frac{k!}{i!\,(k-\cancel{i})!} \cdot {}^{m-k}C_{100-i} \cdot (\cancel{k-i})$$

$$\frac{1}{{}^mC_{100}\,(m-100)} \sum_{i=0}^{100} \frac{k(k-1)!}{i!\,(k-i-1)!} \; {}^{m-k}C_{100-i}$$

$${}^{k-1}C_i$$

$$\frac{k}{{}^mC_{100}\,(m-100)} \sum_{i=0}^{100} \frac{(k-1)!}{i!\,(k-1-i)!} \; {}^{m-k}C_{100-i}$$

$$\frac{k}{{}^mC_{100}\,(m-100)} \sum_{i=0}^{100} {}^{k-1}C_i \; {}^{m-k}C_{100-i} \rightarrow \text{Hypergeometric Sum}$$

$$\frac{k}{{}^{m}C_{100}(m-100)} \cdot {}^{m-1}C_{100}.$$

HW

Q. The coeff of $x^4$ in the expansion of $(1+x-2x^2)^7$ is :-

(a) 81 (b) 91 (c) $-81$ (d) $-91$.