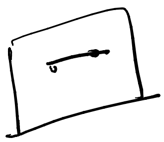
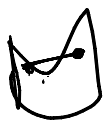
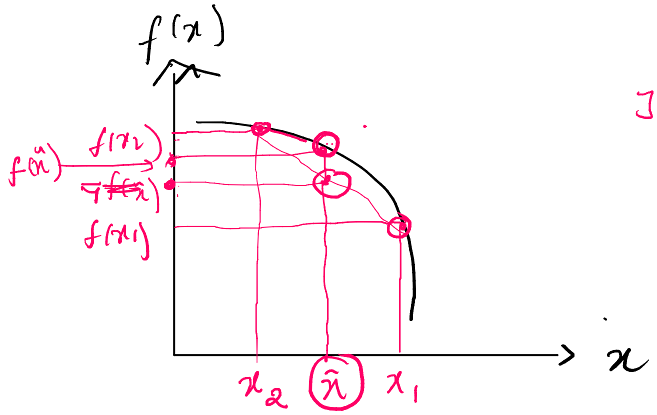
02 April 2024 07:12 PM

Concave, convex and Quasiconcave, Quasiconvex functions.

$$\lambda x_1 + (1-\lambda) x_2$$







There is an interval (a,b) $x_1, x_2 \in I$

Convex combination of x_1 and x_2 is given by

$$\hat{x} = \lambda x_1 + (1-\lambda) x_2$$

for any $\lambda \in [0,1]$

By cross sectional formula

$$\bar{x} = \frac{\lambda x_1 + (1-\lambda) x_2}{\lambda + (1-\lambda)}$$

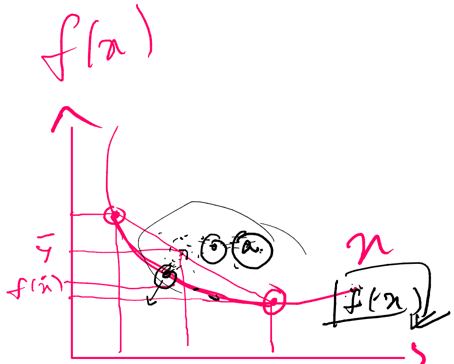
$$\bar{y} = \frac{\lambda f(x_1) + (1-\lambda) f(x_2)}{\lambda (1-\lambda)}$$

So the cordinates are $\left(\lambda x_1 + (1-\lambda) x_2 , \lambda f(x_1) + (1-\lambda) f(x_2) \right)$

$$f(\hat{x}) > \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$\boxed{f(\lambda x_1 + (1-\lambda) x_2) > \lambda f(x_1) + (1-\lambda) f(x_2)}$$

Is strictly concave function



$$\boxed{f(\lambda x_1 + (1-\lambda) x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)}$$

Is strictly convex function

Let f be a multivariate function defined on a set S.
We say that f is quasiconcave if, for any number 'a', the set of points for which $f(x) \geq a$ is convex

\# Let f be a function of many variables defined on the set S. For any real number a, the set

$$P_a = \{x \in S : f(x) \geq a\}$$

is called the upper level set of f for a.

(at least equal to a)

\# Let f be the function of many variables defined on the set S. For any real number a, the set $P^a = \{x \in S : f(x) \leq a\}$

is called the lower level set of f for a.

(at most a)

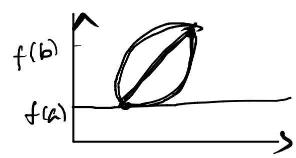
The function f of many variables defined on a convex set S is quasiconvex if every lower level set of f is convex set → is

lower level set of f :

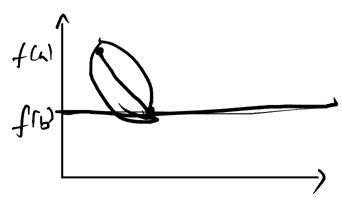
Quasiconcave : $f[(1-\lambda)a + \lambda b] \geq \min[f(a), f(b)]$

Then f is called quasiconcave

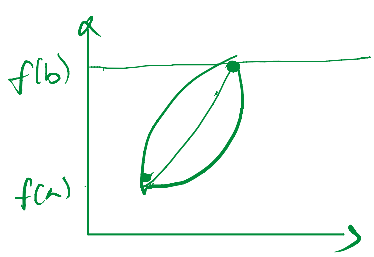
when $f(a) < f(b)$



$f(b) < f(a)$



Convex : $f[(1-\lambda)a + \lambda b] \leq \max[f(a), f(b)]$



$f(a) < f(b)$

from here we get: Every concave fn is Quasiconcave
Every convex fn is Quasiconvex
but converse is not true.

but converse
is not true.

$F(g(x))$ is quasiconcave iff $g(x)$ is quasiconcave & F is strictly increasing.

$F(g(x))$ is quasiconvex iff $g(x)$ is quasiconvex & F is strictly increasing.

Q) $z = e^{-x^2}$. Prove whether quasiconcave or not.

Let $g(x) = -x^2 \Rightarrow$ concave $\Rightarrow$ Quasiconcave.

$F(u) = e^u$ F is strictly incr.

$z = f(g(x)) \Rightarrow z$ is quasiconcave

concave + concave = concave
concave − convex = concave
convex − concave = convex.

Q) $z = A x^{\alpha} y^{\beta} \Rightarrow$ Quasiconcave or not.

$\ln z = \ln A + \alpha \ln x + \beta \ln y$

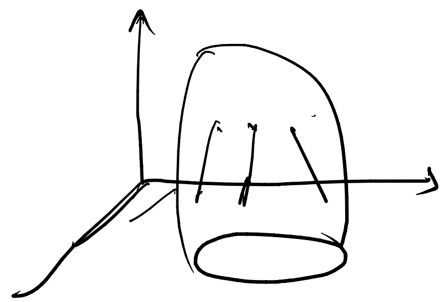
$\ln z$ is concave $\Rightarrow$ Quasiconcave.
$z = e^{\ln z}$

$g(z) = \ln z$

$f(g(z)) = e^{g(z)}$ is strictly incr.

$z = f(x,y) = xy$ if $xy \geq 2$ prove that it is quasiconcave.

Is it quasiconvex as well?



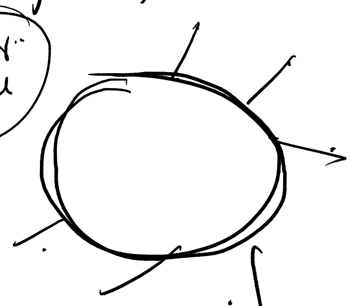
$xy \leq 2 \Rightarrow$ it is not a quasiconvex fn.

$f(x,y) = x^2+y^2$ $x, y > 0$. lower level is 4.

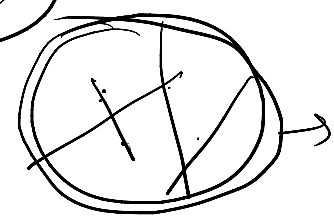
$x^2+y^2 \geq 4$

upper level

$x^2+y^2 \leq 4$



Not a convex set $\Rightarrow$ Not Quasiconcave



$\rightarrow$ Convex set $\Rightarrow$ quasi convex

$y = x$

Hessian Matrix

$$f\,(x_1, x_2)$$

$$f : R^n \to R$$

$$H_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & & \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$z = f\,(x, y)$$

$$H_{(x,y)} = \begin{bmatrix} \dfrac{\partial^2 z}{\partial x^2} & \dfrac{\partial^2 z}{\partial x \partial y} \\ \dfrac{\partial z}{\partial y \partial x} & \dfrac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

Prop

(i) Positive definite if $\left.\begin{array}{l} D_1 > 0 \\ D_2 > 0 \\ D_3 > 0 \end{array}\right\}$ principle minors

(ii) Negative definite if

$D_1 < 0 \Rightarrow$ first is -ve

$D_2 > 0$

$D_3 < 0$

remaining minors should have alternative sign.

(c) positive semi-definite

$D_1 > 0$

$D_2 \geq 0$

$D_3 \geq 0$

(d) negative semi-definite

$D < 0$

$D_2 \geq 0$

$D_3 \leq 0$

$$f(x) = x_1^2 + x_2^2 + x_1 + x_2 - 1$$

$\frac{\partial f}{\partial x_1} = 2x_1 + 1$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

$$\left.\begin{aligned} D_1 &= |2| = 2 > 0 \\ D_2 &= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \end{aligned}\right\}$$ +ve definite
strictly convex
↓
min value.

$$f(x) = 2x_1^2 - x_2^2 + x_1 x_2 + 3x_1 + 4x_2 + 17.$$

Try : 1) $f(x) = x_1^3 - x_2^2 + 3x_2 + 10$ at point $(0,15)$ and $(2,2)$

2) $f(x) = 2x_1^2 - x_2^2 + x_1 x_2 + 3x_1 + 4x_2 + 17$

3) $f(x) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1 x_2 - 2x_2^2 + 41$

4) $f(x) = 3e^{2x_1+1} + 2e^{x_2+5}$ find the definiteness only.

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