R.S: $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ => used to construct the C.I for $\bar{X}$, s.e($\bar{X}$), ...

Sample Mean

C.I for $\bar{X}$: $P\left[-\tau_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq \tau_{\alpha/2}\right] = (1-\alpha)$.

s.e of $\bar{X}$: $\sqrt{Var(\bar{X})} = \frac{\hat{\sigma}}{\sqrt{n}}$

Sample Median [we don't know the sampling distribution].

Note: For sample measures which do not follow a standard theoretical prob distn, we cannot compute the CI & s.e directly. Here we need bootstrap.

Bootstrapping: [Eg: Sample Mean ($\bar{x}$) / Sample Median ($\tilde{x}$)]

$X_1, X_2, \dots, X_n \overset{iid}{\sim} f(\mu, \sigma^2)$ [sample size = n].

(i) Construct new samples by performing sampling with replacement on the existing sample [bootstrap samples of size n]

Eg:

	U_1	U_2	...	U_n	Sample Mean
BS_1	X_3	X_1	...	X_1	→ $\bar{x}_1$
BS_2	X_2	X_2	...	X_5	→ $\bar{x}_2$
⋮					⋮
BS_{1000}	X_3	X_3	...	X_1	→ $\bar{x}_{1000}$

→ Distribution of sample Mean values. [approx sampling distribution].

[Boostrap distribution]

Eg: (original) Sample: 1, 2, 3 ..=> (original) Sample ...

...ibution]

Eg: (original) Sample: 1, 2, 3 $\Rightarrow$ (original) Sample Mean = 2.   [Boostrap distribution]

No. of bootstrap samples $= 3^3 = 27$.

	U_1	U_2	U_3
BS_1 :	1	1	1
BS_2 :	1	1	2
BS_3 :	1	1	3
BS_4 :	1	2	1
BS_5 :	1	2	2
BS_6 :	1	2	3

Avg sample mean $\bar{x} = \frac{1}{1000}\sum_{i=1}^{1000} \bar{x}_i \Rightarrow$ [Bootstrap estimate].

Variance of sample mean $= \frac{1}{1000-1}\sum_{i=1}^{1000}(\bar{x}_i - \bar{x})^2$

$$\therefore \ s.e(\bar{x}) = \sqrt{\frac{1}{1000-1}\sum_{i=1}^{1000}(\bar{x}_i - \bar{x})^2}$$

[Bootstrapped standard error].

In general, for any test statistic $\hat{\theta}$, the $s.e(\hat{\theta})$ computed based on 'B' boostrap samples will be given by:

$$\bar{\hat{\theta}} = \frac{1}{B}\sum_{i=1}^{B}\hat{\theta}_i$$

$$\boxed{s.e(\hat{\theta}) = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\hat{\theta}_i - \bar{\hat{\theta}})^2}}$$

(Here $B \leq n^n$, n = original sample size)

$i-1$ sample size)

Q. Compute the C·I for $\hat{\theta}$:

Consider the small sample testing criteria for sample mean:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim \tau \quad \Rightarrow \text{ If } \sigma \text{ is unknown, we replace } \sigma \text{ by } s'$$

$$T = \frac{\bar{x}-\mu}{s'/\sqrt{n}} \sim t_{(n-1)}$$

[In case of Bootstrap., as popln variance is known, the t-distn app is more relevant] .

construct the C·I:

$$P\left[-t_{\alpha/2;(n-1)} \leq \frac{\bar{x}-\mu}{se(\bar{x})} \leq t_{\alpha/2;(n-1)}\right] = (1-\alpha) .$$

$$\Rightarrow P\left[\bar{x} - t_{\alpha/2;(n-1)}\, s.e(\bar{x}) \leq \mu \leq \bar{x} + t_{\alpha/2;(n-1)}\, s.e(\bar{x})\right] = (1-\alpha)$$

For bootstrap replace these by bootstrap. estimates.

∴ Bootstrap C·I for θ :-

$$P\left[\bar{\hat{\theta}} - t_{\alpha/2;(n-1)}\, s.e(\hat{\theta}) \leq \theta \leq \bar{\hat{\theta}} + t_{\alpha/2;(n-1)}\, s.e(\hat{\theta})\right] = (1-\alpha)$$

Parametric Bootstrap:

r.s: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F(\theta)$ use bootstrap find an estimate of θ.