A. Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

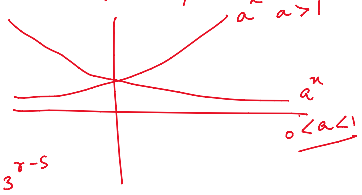
is an integer if

(a) $r + s \leqslant 0$,
(b) $s \leqslant 0$,
(c) $r \leqslant 0$,
(d) $r \geqslant s$.

$6 = 2 \times 3$
$12 = 2^2 \times 3$
$9 = 3^2$
$8 = 2^3$

factorize to prime factors.

x integer. $\neq 0$



$(x^a)^b = x^{ab}$

$x^a \cdot x^b = x^{a+b}$

$\frac{x^a}{x^b} = x^{a-b}$

$$\frac{2^{r+s} \cdot 3^{r+s} \cdot (2^2)^{r-s} \cdot 3^{r-s}}{(2^3)^r \cdot (3^2)^{r+2s}}$$

$$= \frac{2^{r+s+2(r-s)} \cdot 3^{r+\cancel{s}+r-\cancel{s}}}{2^{3r} \cdot 3^{2r+4s}} = \frac{2^{3r-s} \cdot 3^{2r}}{2^{3r} \cdot 3^{2r+4s}}$$

$= 2^{3r-s-3r} \cdot 3^{2r-2r-4s}$

$= 2^{-s} \cdot 3^{-4s}$.

$s \leq 0$

B. The greatest value which the function

$$f(x) = \left(3\sin^2(10x + 11) - 7\right)^2$$

takes, as x varies over all real values, equals

(a) -9, (b) 16, (c) 49, (d) 100.

C. The number of solutions x to the equation

$$7\sin x + 2\cos^2 x = 5,$$

in the range $0 \leqslant x < 2\pi$, is

(a) 1, (b) 2, (c) 3, (d) 4.

$7\sin x + 2 - 2\sin^2 x = 5$

$2\sin^2 x - 7\sin x + 3 = 0$

$2\sin^2 x - 6\sin x - \sin x + 3 = 0$

$2\sin x(\sin x - 3) - 1(\sin x - 3) = 0$

$(2\sin x - 1)(\sin x - 3) = 0$

D. The point on the circle

$$(x-5)^2 + (y-4)^2 = 4$$

which is closest to the circle

$$(x-1)^2 + (y-1)^2 = 1$$

is

(a) $(3.4, 2.8)$, (b) $(3, 4)$, (c) $(5, 2)$, (d) $(3.8, 2.4)$.

$\vec{r} = \vec{a} + \lambda\vec{b}$

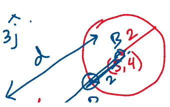
$= (5\hat{i} + 4\hat{j}) + \left(\frac{2}{5}\right)(-4\hat{i} - 3\hat{j})$

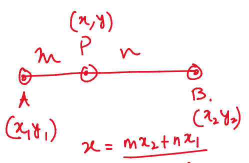
$\vec{BA} = -4\hat{i} - 3\hat{j}$

$c + 2 \times (-4)$

$d = 5$

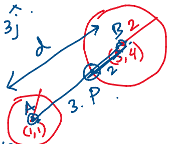
$P\left(\frac{2 \times 1 + 3 \times 5}{5},\right.$





$x = m x_2 + n x_1$

$\vec{BA} = -4\hat{i} - 3\hat{j}$



$x = \frac{5 + 2\times(-4)}{5}$

$= 5 - 8/5 = \frac{17}{5} =$

$P\left(\frac{2\times1 + 3\times5}{5}, \frac{2\times1 + 3\times4}{5}\right)$

$x = \frac{mx_2 + nx_1}{m+n}$

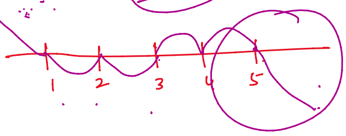
E. If x and n are integers then

$$f(x) = (1-x)^n (2-x)^{2n} (3-x)^{3n} (4-x)^{4n} (5-x)^{5n}$$

is

(a) negative when $n > 5$ and $x < 5$,
(b) negative when n is odd and $x > 5$,
(c) negative when n is a multiple of 3 and $x > 5$,
(d) negative when n is even and $x < 5$.

$n = \text{odd}$ $\quad$ $n = 1$



F. The equation

$$8^x + 4 = 4^x + 2^{x+2}$$

has

(a) no real solutions;
(b) one real solution;
(c) two real solutions;
(d) three real solutions.

$2^x = a$. $\quad 2^{3x} = a^3$.

$a^3 + 4 = a^2 + 4a$.

$2^{x+2} = 2^x \cdot 2^2 = 4\cdot 2^x$

$4^x = 2^{2x} = (2^x)^2 = a^2$

$a^3 - a^2 - 4a + 4 = 0$

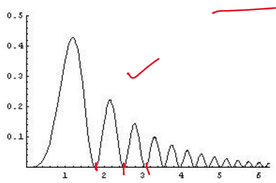
$a^2(a-1) - 4(a-1) = 0$

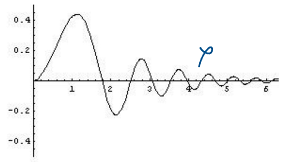
$(a-1)(a+2)(a-2) = 0$

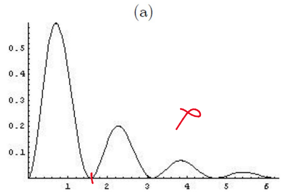
$a = 1, \cancel{-2}, 2$

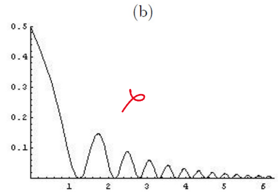
G. On which of the axes below is a sketch of the graph

$$y = 2^{-x}\sin^2(x^2)?$$


(a)


(b)


(c)


(d)

$x = 0$

$y = 0$

$x^2 = 0, \pi, 2\pi, 3\pi, \ldots$

$x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}$

1.7 $\quad$ 2.4 $\quad$ 3.1

H. Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,\mathrm{d}x + \int_1^2 2f(x)\,\mathrm{d}x = 7,$$
$$\int_0^2 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x = 1.$$

It follows that $\int_0^2 f(x)\,\mathrm{d}x$ equals

(a) -1, (b) 0, (c) $\frac{1}{2}$, (d) 2.

$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx.$

$A \quad + \quad B$

$3A+2B=7$

$A+B+B=1$

$A+2B=1$

$2A=6$

$A=3$

$2B=-2$

$B=-1.$

I. Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1,$$

0

then the greatest possible value of a is

(a) $\frac{1}{10}$, (b) 1, (c) $\sqrt{10}$, (d) $10^{\sqrt{2}}$.

$x+y=1$

max min

$(\log a)^2 = \frac{1}{4}$

$\log_{10} a = \frac{1}{2}$

$a = 10^{1/2}$

J. The inequality

$$(n+1)+(n^4+2)+(n^9+3)+(n^{16}+4)+\cdots+(n^{10000}+100) > k$$

is true for all $n \geqslant 1$. It follows that

(a) $k < 1300$,
(b) $k^2 < 101$,
(c) $k \geqslant 101^{10000}$,
(d) $k < 5150$.

LHS is an increasing function with n.

$n=1$

$2+3+4+\cdots+101$

$= \frac{100}{2}(2+101) = 51.5\times 100$

$= 5150$

Let

$$f_n(x) = (2 + (-2)^n)\,x^2 + (n+3)\,x + n^2$$

where n is a positive integer and x is any real number.

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.

For what values of n does $f_n(x)$ have a maximum value (as x varies)?

[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$.

Find an expression, simplified as much as possible, for

$$f_1(f_1(f_1(\cdots f_1(x))))$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2(f_2(f_2(\cdots f_2(x)))),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

squares 1 to 32

$f_3(x) = -6x^2 + 6x + 9$

$f_3(x)_{max} = -6 \times \frac{1}{4} + 6 \times \frac{1}{2} + 9$

$= 12 - \frac{3}{2} = \frac{21}{2}$

$(a+b)^n = a^n + na^{n-1}b$

$\sqrt{42} = (36+6)^{1/2} = 36^{1/2} + \frac{1}{2}(36)^{-1/2} \cdot 6$

$= 6 + \frac{6}{2 \times 6}$

$\sqrt[3]{42} = (27+15)^{1/3}$

$= 27^{1/3} + \frac{1}{3} \cdot 27^{-2/3} \cdot 15$

$= 3 + \frac{\cancel{15}^{5}}{\cancel{3} \times 9} = 3 + 0.5$

$\sqrt{42} = 6 + \frac{6}{2 \times 6}$

$= 6 + \frac{1}{2} = 6.5$

36

$\sqrt{590} = 24 + \frac{14}{2 \times 24}$

576

$24 + \frac{7}{24}$

$= 24 + 0.3$

105×112

+5 +12

11760

17% × 260

10% + 5% + 2%

88 × 96

−12 −4

8448

95 × 102

−5 +2

9700

− 10

9690

1 to 10

88 × 96

86 × 87

−14 −13

7382

7482

Let

$$I(c) = \int_0^1 \left((x-c)^2 + c^2\right)\,\mathrm{d}x$$

where c is a real number.

(i) Sketch $y = (x-1)^2 + 1$ for the values $-1 \leqslant x \leqslant 3$ on the axes below and show on your graph the area represented by the integral $I(1)$.

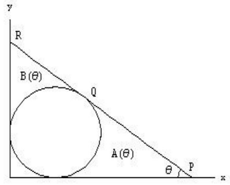
(ii) Without explicitly calculating $I(c)$, explain why $I(c) \geqslant 0$ for any value of c.

(iii) Calculate $I(c)$.

(iv) What is the minimum value of $I(c)$ (as c varies)?

(v) What is the maximum value of $I(\sin\theta)$ as θ varies?

In the diagram below is sketched the circle with centre $(1,1)$ and radius 1 and a line L. The line L is tangential to the circle at Q; further L meets the y-axis at R and the x-axis at P in such a way that the angle OPQ equals θ where $0 < \theta < \pi/2$.



(i) Show that the co-ordinates of Q are

$$(1 + \sin\theta, 1 + \cos\theta),$$

and that the gradient of PQR is $-\tan\theta$.

Write down the equation of the line PQR and so find the co-ordinates of P.

(ii) The region bounded by the circle, the x-axis and PQ has area $A(\theta)$; the region bounded by the circle, the y-axis and QR has area $B(\theta)$. (See diagram.)

Explain why

$$A(\theta) = B(\pi/2 - \theta)$$

for any θ.

Calculate $A(\pi/4)$.

(iii) Show that

$$A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}.$$