Eg: Consider $N(\mu, \sigma^2)$ [$N(\mu, 1)$] popln where $\sigma^2$ is known.

To test: $H_0: \mu = \mu_0$ vs $H_1: \mu = \mu_1$ $[\mu_1 > \mu_0]$.

Consider r.s $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ [$\sigma^2$ is known].

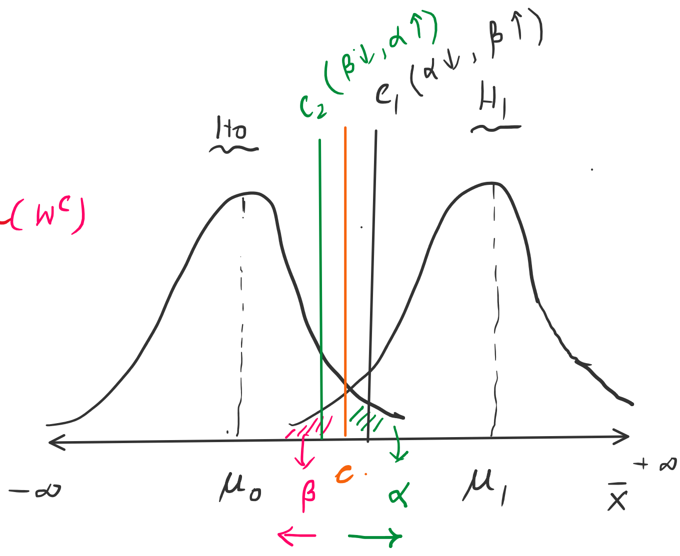
Compute sample mean $\bar{x}$

Fix a critical pt 'c'. If $\bar{x} > c \Rightarrow$ Reject $H_0$. } Testing criteria
If $\bar{x} < c \Rightarrow$ Accept $H_0$.

2 Possible Errors.

Type I Error $\alpha = P[\bar{x} > c \mid H_0]$. ($\bar{x} > c$ → Critical Region ($W$))

Type II Error $\beta = P[\bar{x} < c \mid H_1]$. ($\bar{x} < c$ → Acceptance Region ($W^c$))




Here $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

Under $H_0$: $\bar{x} \sim N(\mu_0, \frac{\sigma^2}{n})$

Under $H_1$: $\bar{x} \sim N(\mu_1, \frac{\sigma^2}{n})$.

We cannot obtain any critical pt 'c' that minimizes both $\alpha, \beta$.
We will fix the level of $\alpha$ at at a pre-determined level and then try to minimize $\beta$ [or max $(1-\beta)$].

Pre-determined level of $\alpha \Rightarrow$ Level of significance of the Test
Usually we choose L.o.s as 0.05 (5%) / 0.01 (1%)

Here: $\beta = P[\bar{x} < c \mid H_1]$ (Accept $H_0$ when $H_0$ is false)

$(1-\beta) = P[\bar{x} \geq c \mid H_1]$ (Rejecting $H_0$ when $H_0$ is false)

$\Rightarrow$ Prob of tak...

$(1-\beta) = P[x \geq c \mid H_1]$ (Rejecting $H_0$ when $H_0$ is false)

$\Rightarrow$ Prob of taking the correct decision from the Test (Power of the Test)

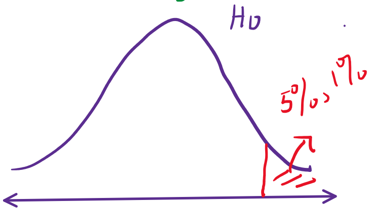
## Most Powerful Test (MP Test):

Obj: Keeping '$\alpha$' at a pre-determined level, max $(1-\beta)$.

To test $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$.

The critical region $W$ is the most powerful critical region of size $\alpha$ if:

$$P(x \in W / H_0) = \alpha = P(x \in W_1 / H_0).$$




$$\underbrace{P(x \in W / H_1)}_{\text{power of test under critical region } W} \geq \underbrace{P(x \in W_1 / H_1)}_{\text{power of the test under any other critical region } W_1} \quad \forall\, W_1.$$

## Neyman-Pearson Lemma (NP Lemma):

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x, \theta)$.

To test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$.

$\therefore$ Likelihood fn $L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta)$.

$\therefore$ If $\exists$ a critical region $W$ and a constant $k > 0$ s.t

$$\frac{L(x, \theta_1)}{L(x, \theta_0)} \geq k \;\; \forall\, x \in W \quad \& \quad \frac{L(x, \theta_1)}{L(x, \theta_1)} < k \;\; \forall\, x \in W^c$$

Then $W$ is the best critical region for performing the given test.

HW.

Q. Use the NP lemma to find the best critical region to test: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 > \theta_0]$ for $N(\theta, \sigma^2)$ popln where $\sigma^2$ is known. Hence find the power of the test.

r.s $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ $[\sigma^2$ is known$]$.

$$f(x_i, \theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \theta}{\sigma}\right)^2}$$

$$\therefore L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}.$$

NP: $$\frac{L(x, \theta_1)}{L(x, \theta_0)} \geq K \quad \forall x \in W.$$

$$\Rightarrow \frac{e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta_1)^2}}{e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta_0)^2}} \geq K.$$