LINEAR ALGEBRA SPECIAL SESSION

Thursday, November 2, 2023 7:32 PM

7:30-9:30

Continuous Eval

Neighbourhood

Trace

Mat

$Tr(AB) = Tr(BA)$

A sq matrix (n) order $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

R(A) is less than (n-1) Adj (A) => 0 9062395123

R(A) n = (n-1) Adj (A) => 1

R(A) n (n) Adj (A) = (n)

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq 0$

R = 2 Det

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $R \neq 2$

$\rho(AB) \geq \rho(A) + \rho(B) - n$

$A_{m \times n}$, $B_{n \times k}$

$X^T A X$ → n variables

Positive definite → All E.V. > 0

Negative definite → All EV < 0

" Semi-definite all are EV non-positive EV ≤ 0

Positive semi-def → EV ≥ 0

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Linear Algebra Q...

R + N = d

1. *Consider the subspace* $W = \{[a_{ij}] : a_{ij} = 0 \text{ if } i \text{ is even}\}$ *of all* 10×10 *real matrices. Then the dimension of W is* **(1 mark)**

(A) 25 (B) 50 (C) 75 (D) 100

$a_{ij} = 0$ for i = even

$a_{2j} = 0$ $a_{4j} = 0$ $a_{10j} = 0$ $\forall j = 1, 2 \ldots 10$

50 dimly indept condn

dim W = 100 − 50 = 50

2. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear map satisfying $T(e_1)=e_2$, $T(e_2)=e_3$, $T(e_3)=0$, $T(e_4)=e_3$, where $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$. Then **(2 marks)**

(A) T is idempotent (B) T is invertible (C) Rank $T = 3$ (D) T is nilpotent

4. *For any $n \in \mathbb{N}$, let P_n denote the vector space of all polynomials with real coefficients and of degree at most n. Define $T: P_n \to P_{n+1}$ by $T(p(x)) = p'(x) - \int_0^x p(t)dt$. Then the dimension of the null space of T is*

(2 marks)

(A) 0 *(B) 1* *(C) n* *(D) n+1*

6. *The number of linearly independent eigenvectors of the matrix* $\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ *is* **(2 marks)**

(A) 1 (B) 2 (C) 3 (D) 4

Statement for linked Answer Questions 7 and 8:

Let $N = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$NN^t = N^t N = I$ → orthogonal.

$$|M - \lambda I| = (2-\lambda)(1-\lambda)(3-\lambda)(4-\lambda) - 2.2.(3-\lambda)(4-\lambda)$$

$$= (3-\lambda)(4-\lambda)\left[(2-\lambda)(1-\lambda) - 4\right]$$

$$= (3-\lambda)(4-\lambda)(\lambda^2 - 3\lambda - 2)$$

$$= (3-\lambda)(4-\lambda)\left(\lambda - \frac{3+\sqrt{17}}{2}\right)\left(\lambda - \frac{3-\sqrt{17}}{2}\right)$$

$$\lambda = 3, 4, \frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}$$

$1+\omega+\omega^2=0$

$\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)=A$

7. *Then N is*

(A) non-invertble *(B) skew-symmetric* *(C) symmetric* *(D) orthogonal* **(2 marks)**

$tr(NMN^T) = TR(NN^TM)$

$TR(AB) = TR(BA)$

$TR(IM) = TR(M)$

$$\begin{matrix} 3/5 & -4/5 & 0 \\ 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$N^T N = I$

8. *If M is any 3×3 real matrix, then trace (NMN^{t}) is equal to* **(2 marks)**

(A) $[trace(N)]^2$ trace (M) (B) 2 trace(N) + trace(M)

(C) trace(M) (D) $[trace(N)]^2$ + trace(M)

10. *The minimal polynomial associated with the matrix* $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ *is* **(1 mark)**

(A) $x^3 - x^2 - 2x - 3$ *(B)* $x^3 - x^2 + 2x - 3$ *(C)* $x^3 - x^2 - 3x - 3$ *(D)* $x^3 - x^2 + 3x - 3$

Chhm short cut — Since A → Companion Matrix
So characheric Polynomial of A = minimal polynomial of A
i.e. $C_A(x) = m_A(x)$
$m_A(x)$ is given by $x^3 - x^2 - 2x - 3$

11. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$, *then the trace of* A^{102} *is* ***(2 marks)***

(A) 0 *(B) 1* *(C) 2* *(D) 3*

Ev of A $1, \omega, \omega^2$

$A^{102} \to 1^{102}, \omega^{102}, \omega^{204}$

1 1 1 3

12. *Which of the following matrices is not diagonalizable?* (2 marks)

(A) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ (C) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

EV: 1, 1

If a sq matrix has all E.V. are equal + diagonalizable → diagonal matrix.

(D)

14. *The dimension of the range space of* T^2 *is* **(2 marks)**

(A) 0 *(B) 1* *(C) 2* *(D) 3*

$x^2+2x+1=0$

15. *The dimension of the null space of* T^3 *is* **(2 marks)**

(A) 0 *(B) 1* *(C) 2* *(D) 3*

16. *If the nullity of the matrix* $\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$ *is 1, then the value of 'k' is* **(1 mark)**

(A) -1 (B) 0 (C) 1 (D) 2

$R_2 - kR_1 \quad R_3 - R_1$

$$\begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ k & 1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1+k & 2+2k \\ 0 & 2 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 1+k & 2(1+k) \end{bmatrix}$$

R = No of non-zero Rows..

for nullity = 1

R = 2

k + 1 = 0

k = -1

17. *If a* 3×3 *real skew-symmetric* **matrix** *has an eigenvalue* $2i$, *then one of the remaining eigenvalues is* **(1 mark)**

(A) $\frac{1}{2i}$ (B) $-\frac{1}{2i}$ (C) 0 (D) 1

EV → Either Purely Imaginary or Zero

Imaginary Roots → Pairs

Eigenvalue $2i$ $-2i, 0$

As, odd Skew symmetric matrix has det = 0

So, one of them has to be Zero

As, det = $\prod$(E.V)

19. *Let* $T : P_3[0,1] \to P_2[0,1]$ *be defined by* $(Tp)(x) = p''(x) + p'(x)$. *Then the matrix representation of* T *with respect to the basis* $\{1, x, x^2, x^3\}$ *and* $\{1, x, x^2\}$ *of* $P_3[0,1]$ *and* $P_2[0,1]$ *respectively is*

(2 marks)

21. *The distinct eigenvalues of the matrix* $\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{bmatrix}$ *are* **(1 mark)**

(A) 0 and 1 *(B) 1 and -1* *(C) 1 and 2* *(D) 0 and 2*

$\Rightarrow -\lambda\left[1+\lambda^2-2\lambda-1\right]=0$

$\lambda = 0, 2$

22. *The minimal polynomial of the matrix* $\begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ *is* **(1 mark)**

(A) $x(x-1)(x-6)$ (B) $x(x-3)$ (C) $(x-3)(x-6)$ (D) $x(x-6)$

Block diagonal matrix

char(A) = ch(B) × ch(C)

= $(x-6) \times x(x-6)$

choice for mini polynomial = $x(x-6)$ or $x(x-6)^2$

As, $(x)(x-6)^2$ is not true

$x(x-6)$

25. *Let* $T:\mathbb{R}^4 \to \mathbb{R}^4$ *be defined by* $T(x, y, z, w) = (x + y + 5w, x + 2y + w, -z + 2w, 5x + y + 2z)$. *The dimension of the eigenspace of T is* **(2 marks)**

(A) 1 (B) 2 (C) 3 (D) 4

Statement for linked Answer Questions 26 and 27:

The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ *can be decomposed into the product of a lower triangular matrix L and an upper triangular matrix U as A=LU, where*

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

. Let $x, z \in \mathbb{R}^3$ *and* $b = [1, 1, 1]^T$

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$Ux = z$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$\therefore x = (2, 1, -2)^T$

$Ux = z$

27. *The solution $x=[x_1, x_2, x_3]^T$ of the system $U x=z$ is* ***(2 marks)***

(A) $[2,1,-2]^T$ (B) $[2,1,2]^T$ (C) $[-2,-1,-2]^T$ (D) $[-2,1,-2]^T$

28. *Let* $\alpha = e^{2\pi i/5}$ *and the matrix* $M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$. *Then the trace of matrix* $I+M+M^2$ *is* **(1 mark)**

(A) -5 *(B) 0* *(C) 3* *(D) 5*

30. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *then* A^{50} *is* **(1 mark)**

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 50 & 0 & 0 \\ 50 & 0 & 1 \end{bmatrix}$ *(B)* $\begin{bmatrix} 1 & 0 & 0 \\ 48 & 0 & 0 \\ 48 & 0 & 1 \end{bmatrix}$ *(C)* $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$ *(D)* $\begin{bmatrix} 1 & 0 & 0 \\ 24 & 1 & 0 \\ 24 & 0 & 1 \end{bmatrix}$

35. *The possible set of eigen values of a* 4×4 *skew – symmetric orthogonal real matrix is* ***(1 mark)***

(A) $\{\pm i\}$ (B) $\{\pm i, \pm 1\}$ (C) $\{\pm 1\}$ (D) $\{0, \pm i\}$

36. *Let P be a* 2×2 *complex matrix such that trace(P)=1 and det(P)=-6. Then, trace of* (P^4-P^3) *is* ________

(1 mark)

38. *Let M be the real vector space of* 2×3 *matrices with real entries. Let* $T:M\longrightarrow M$ *be defined by*

$$T\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}\right)=\begin{bmatrix} -x_6 & x_4 & x_1 \\ x_3 & x_5 & x_2 \end{bmatrix}$$

. The determinant of T is ______ ***(2 marks)***

41. *Let X be the space of all* 4×3 *matrices with entries in the field of three elements. Then the number of matrices of rank three in M is* ***(2 marks)***

(A) $(3^4 - 3)(3^4 - 3^2)(3^4 - 3^3)$

(B) $(3^4 - 1)(3^4 - 2)(3^4 - 3)$

(C) $(3^4 - 1)(3^4 - 3)(3^4 - 3^2)$

(D) $3^4(3^4 - 1)(3^4 - 2)$

42. Let V be a vector space of dimension $m \geq 2$. Let $T: V \rightarrow V$ be linear transformation such that $T^{n+1} = 0$ and $T^n \neq 0$ for some $n \geq 1$. Then which of the following is necessary TRUE? **(2 marks)**

(A) Rank$(T^n) \leq$ Nullity (T^n)

(B) trace$(T) \neq 0$

(C) T is diagonalizable

(D) $n=m$

43. Let $A \in M_3(\mathbb{R})$ be such that $\det(A-I)=0$, where I denotes the 3×3 identity matrix. If trace $(A)=13$ and $\det(A)=32$, then the sum of squares of the eigen values of A is ______

(1 mark)

49. *Let* $T_1, T_2 : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ *be linear transformations such that* $rank(T_1) = 3$ *and* $nullity(T_2) = 3$.
Let $T_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ *be a linear transformation such that* $T_3 \circ T_1 = T_2$. *Then* $rank(T_3)$ *is* _________

(2 marks)

52. *Let M be a* 3×3 *matrix and suppose that 1,2 and 3 are the eigenvalues of M. If* $M^{-1}=\frac{M^2}{\alpha}-M+\frac{11}{\alpha}I_3$ *for some scalar* $\alpha\neq 0$*, then* α *is equal to*_________. ***(1 mark)***

53. *Let M be a* 3×3 *singular matrix and suppose that 2 and 3 are eigenvalues of M. Then the number of linearly independent eigenvectors of* $M^3 + 2M + I_3$ *is equal to*____________. ***(1 mark)***

54. *Let M be a* 3×3 *matrix such that* $M\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}6\\-3\\0\end{pmatrix}$ *and suppose that* $M^3\begin{pmatrix}1\\-\frac{1}{2}\\0\end{pmatrix}=\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ *for some*

57. Let M be an invertible Hermitian matrix and let $x, y \in \mathbb{R}$ be such that $x^2 < 4y$. Then. **(2 marks)**

(A) both $M^2 + xM + yI$ and $M^2 - xM + yI$ are singular

(B) $M^2 + xM + yI$ is singular but $M^2 - xM + yI$ is non-singular

(C) $M^2 + xM + yI$ is non-singular but $M^2 - xM + yI$ is singular

(D) both $M^2 + xM + yI$ and $M^2 - xM + yI$ are non-singular

71. *Let* $A = (a_{ij})$ *be a* 10×10 *matrix such that* $a_{ij} = 1$ *for* $i \neq j$ *and* $a_{ii} = \alpha + 1$, *where* $\alpha > 0$. *Let* λ *and* μ *be the largest and the smallest eigenvalues of* A, *respectively. If* $\lambda + \mu = 24$, *then* α *equals*____

(2 marks)

72. Let $A = \begin{bmatrix} a & 2f & 0 \\ 2f & b & 3f \\ 0 & 3f & c \end{bmatrix}$, *where a, b, c, f are real numbers and $f \neq 0$. The geometric multiplicity of the largest eigenvalue of A equals _________ .* ***(1 marks)***

73. *Consider the subspaces* $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 + 2x_3\}$

$W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 3x_2 + 2x_3\}$

of $\mathbb{R}^3$. *Then the dimension of* $W_1 + W_2$ *equals* ________. ***(1 marks)***

78. *Consider the matrix* $A = I_9 - 2u^T u$ *with* $u = \frac{1}{3}[1, 1, 1, 1, 1, 1, 1, 1, 1]$, *where* I_9 *is the* 9×9 *identity matrix and* u^T *is the transpose of* u. *If* λ *and* μ *are two distinct eigenvalues of* A, *then* $|\lambda - \mu| =$ ______.

(2 marks)

79. *If the characteristic polynomial and minimal polynomial of a square matrix A are* $(\lambda-1)(\lambda+1)^4(\lambda-2)^5$ *and* $(\lambda-1)(\lambda+1)(\lambda-2)$, *respectively, then the rank of the matrix A + I is* _____, *where I is the identity matrix of appropriate order.* ***(1 mark)***

82. *Let V be the vector space of all 3 × 3 matrices with complex entries over the real field. If* $W_1 = \{A \in V : A = \overline{A}^T\}$ *and* $W_2 = \{A \in V : \text{trace of } A = 0\}$*, then the dimension of* $W_1 + W_2$ *is equal to______.* **(2 marks)**

($\overline{A}^T$ *denotes the conjugate transpose of A)*

86. *Suppose V is a finite dimensional non-zero vector space over $\mathbb{C}$ and $T : V \rightarrow V$ is a linear transformation such that Range(T) = Null space (T). Then which of the following statements is FALSE?* **(2 marks)**

(A) The dimension of V is even

(B) 0 is the only eigenvalue of T

(C) Both 0 and 1 are eigenvalues of T

(D) $T^2 = 0$