STAT 25/1/24

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$$Y = (\alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u_i)$$

MLE

likelihood function

$$L = f(\ldots)$$

$$\frac{\partial L}{\partial x_1}$$

FOC

SOC

$$\frac{\partial L}{\partial x_2}$$

$$L(\theta) = \prod f(x_i, \theta)$$

Maximize for all distributions

$n \quad (x_1, x_2 \ldots x_n) \qquad Poi(\mu)$

$$L(\mu) = \prod_{i=1}^{n} \frac{e^{-\mu} \cdot \mu^{x_i}}{x_i!} = Const \times e^{-n\mu} \cdot \mu^{\sum x_i}$$

$$\log_e L(\mu) = Const - n\mu + \sum x_i \ln \mu_i$$

$$\frac{d}{d\mu} \ln L(\mu) = -n + \frac{\sum x_i}{\mu} = 0$$

$$\bar{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{d^2}{} \ln L(\mu) = -\frac{\sum x_i}{} < 0 = \boxed{max}$$

$$\frac{d^2}{d\mu^2} \ln L(\mu) = -\frac{\sum x_i}{\mu^2} < 0 \Rightarrow \boxed{\text{max}}$$

1 Parameter

estimate is $\overline{X}$

$\boxed{\text{2 Parameter Case}}$

MLE of (μ, σ)

$$L(\mu,\sigma) = \prod \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^{\boxed{2}}\right]$$

$$= \sigma^{-n} \exp\left[-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right] \times \text{Con.}$$

$$\log L(\mu,\sigma) = -n \ln \sigma - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2 + \text{Const}$$

$$\frac{\partial}{\partial\mu} \ln L = -\frac{1}{2\sigma^2}\sum 2(x_i-\mu) = \frac{1}{\sigma^2}\left[\sum x_i - n\mu\right]$$

$$\frac{1}{\partial\mu} \ln L = -\frac{n}{\sigma} - \frac{1}{2}\left(-\frac{2}{\sigma^3}\right)\sum(x_i-\mu)^2$$

$$= \frac{1}{\sigma}\left[\frac{1}{\sigma^2}\sum(x_i-\mu)^2 - n\right] = 0$$

$$\hat{\mu} = \frac{1}{n}\sum x_i = \bar{x}$$

$$\sigma^2 = \frac{n-1}{n} s^2$$

\# ~~exponential~~ X, Y, Z MLE of θ

mean θ 2θ 3θ

$X \sim \exp\left(\frac{1}{\theta}\right)$ $Y \sim \exp\left(\frac{1}{2\theta}\right)$ $Z \sim \exp\left(\frac{1}{3\theta}\right)$

$$\frac{1}{\theta} e^{-\frac{1}{\theta}x} \quad \frac{1}{2\theta} e^{-\frac{1}{2\theta}y} \times \frac{1}{3\theta} e^{-\frac{1}{3\theta}z}$$

$$L(\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \cdot \frac{1}{2\theta} e^{-\frac{y}{2\theta}} \times \frac{1}{3\theta} e^{-\frac{z}{3\theta}}$$

$$= \frac{1}{6}\frac{1}{\theta^3} e^{-\frac{1}{\theta}(x+y/2+z/3)}$$

$$= \text{Con} \times \frac{1}{\theta^3} e^{-\frac{1}{\theta}(x+y/2+z/3)}$$

$$\ln L(\theta) = -3\ln\theta - \frac{1}{\theta}(x+y/2+z/3) + \text{Con}$$

$$\frac{d}{d\theta}\ln L(\theta) = -\frac{3}{\theta} + \frac{1}{\theta^2}(x+y/2+z/3)$$

$$\hat{\theta} = 1/3\,(x+y/2+z/3)$$

ISS 2007

$x_1 = -2$, $x_2 = 1$, $x_3 = 3$, $x_4 = -4$

PDF $\quad f(x|\theta) = \dfrac{\theta^{-x}}{e^{\theta} - e^{-\theta}}$ $\quad -\theta < x < \theta$, $\theta > 0$

Then MLE of θ = ??

$$f(x|\theta) = \frac{e^{-x}}{e^{\theta} - e^{-\theta}} \quad -\theta \le x \le \theta$$

$\theta > 0$

$0 < |x| < \theta$

$$L(\theta) = \prod_{i=1}^{4} \frac{e^{-x_i}}{e^{\theta} - e^{-\theta}}$$

$$\hat{\theta} = \max(|x_i|,\ i=1,2\ldots n) = \max(|-2|, |1|, |3|, |-4|)$$

$$\hat{\theta} = \max(2, 1, 3, 4) = 4$$

$f(x|\theta,\sigma) = \frac{1}{\sigma} e^{-\left(\frac{x_i-\theta}{\sigma}\right)}$ $x > \theta$, $-\infty < \theta < \infty$, $\sigma > 0$

(i) moment method $= 0$, ~~unknown~~.

(ii) MLE

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(x_i - \hat{\theta})}$$

Range $\rightarrow \theta < x_1 < x_2 \dots < x_n < \infty$

$\frac{d}{dx} \rightarrow a \neq \rightarrow \neq 0$ $\rightarrow$ minimum

$$\therefore \hat{\theta} = \boxed{\min}(x_1, x_2 \dots x_n) = X_{(1)}$$

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(x_i - \hat{\theta})}$$

$$L(\sigma) = \frac{1}{\sigma^n} \cdot e^{-\frac{1}{\sigma}\sum(x_i - x_1)}$$

$$\ln L(\sigma) = -n\ln\sigma - \frac{1}{\sigma}\sum(x_i - x_1)$$

$$\frac{d}{d\sigma}\ln L(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum(x_i - X_{(1)})$$

so, $\frac{1}{\sigma}\left[-n + \frac{1}{\sigma}\sum(x_i - X_{(1)})\right] = 0$

$$\hat{\sigma} = \frac{1}{n}\sum(x_i - X_{(1)}) = \bar{X} - X_{(1)}$$

#
$$f(x|\theta) = \theta e^{-\theta x} \quad x>0$$
$$= 0 \quad \text{, otherwise}$$

if $\theta > 0$, then obtain MLE of $P(X>10)$.

$$L(\theta) = \prod \theta \cdot e^{-\theta x}$$
$$= \theta^n \cdot e^{-\theta \sum x_i}$$

(unfinished)

Scenario

\# $X_1, X_2, \ldots X_n$ be a Random Sample

Exp dist $\rightarrow$ mean $= \frac{1}{\lambda}$

MLE of median ??

$X_i \overset{iid}{\sim} \text{Exp}(\lambda) \quad \forall\, i = 1, 2 \ldots n$

$$\text{median} = \text{mean} \ln 2 = \frac{1}{\lambda} \ln 2 \quad \text{— (1)}$$

$$L(\lambda) = \prod \lambda e^{-\lambda x_i}$$
$$= \lambda^n e^{-\lambda \sum x_i}$$

$$\ln L(\lambda) = n \ln\lambda - \lambda \sum x_i$$

$$\frac{d}{d\lambda} \ln L(\lambda) = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

for median $\rightarrow \frac{1}{\lambda} \ln 2 = \frac{1}{\frac{1}{\bar{x}}} \ln 2 = \bar{x} \ln 2$

Assumed value & Parameter Range

mean $\rightarrow$ 2 - 3

$N(\theta, 1) \rightarrow$

If parameter space is $\Theta = [0, 1, 2, 3]$

find MLE $(\theta) = ??$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\theta)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}\sum(x_i-\theta)^2}$$

if $\theta \in (0,1,2,3)$

we estimate $\theta \rightarrow$ when the squared Error is minimum

$$\sum (x_i-\theta)^2 = \sum (x_i - \bar{x} + \bar{x} - \theta)^2$$

$$\Rightarrow \sum (x_i - \bar{x})^2 + \sum (\bar{x}-\theta)^2 + 2\sum (x_i-\bar{x})(\bar{x}-\theta)$$

$\Rightarrow \sum (x_i - \ldots$ $\qquad (\bar{x}-\theta)$

$\Rightarrow \sum (x_i-\bar{x})^2 + n(\bar{X}-\theta)^2 + 2(\bar{x}-\theta)\sum(x_i-\bar{x})$

$\Rightarrow \sum (x_i-\bar{x})^2 + n(\bar{x}-\theta)^2 + 2(\bar{x}-\theta)\ (n\bar{x}-n\bar{x})$

$\Rightarrow \sum (x_i-2.3)^2 + n(2.3-\theta)^2$

the quadratic loss or error is min.

$(2.3-\theta)^2 =$ @ $\underline{\theta = 2.3}$

$X \to Bin(1, p)$

$p \in [1/5, 4/5]$

If the observed value of X is 0

then MLE of p =?

$\hookrightarrow f(x) = p^x(1-p)^{1-x}$

$L(p) = P(X=0)$

$= p^0 (1-p)^{1-0}$

$= p^0 (1-p)^{1-0} = (1-p)$ when $\frac{1}{5} \le p \le \frac{4}{5}$

$L(p)$ attains its max at $\hat{p} = \frac{1}{5}$

T1d

$$f(x|\theta) = \frac{2\theta^2}{x^3} \quad x \geq 0$$

$= 0$

$\theta > 0$ Unknown MLE of θ

Here we can't find MLE by classical method.

$\boxed{\text{Order Statistics}} \rightarrow$ Concept of

Let $L(\theta) = 2^n \theta^{2n} \cdot \frac{1}{\prod_{i=1}^{n} X_i^3}$

Here, value of θ satisfies the range $\{\theta < X_1 < X_2 \ldots X_n < \infty\}$

$\therefore \hat{\theta} = \min\{X_1, X_2 \ldots X_n\} = X_{(1)}$