03 March 2024 08:13 PM

Mathematical Economics

Application of Log and exponential function.

Topic : | Optimal Timing |

A problem of wine storage :

Suppose that a certain wine dealer is in possession of a particular quantity of wine, which he can either sell at the present time $(t=0)$ for a sum of $K or else store for some length of time and then sell at a higher value.

~~$t=0$~~

Storage cost is nil.

The growing value (V) of the wine

$$V = Ke^{\sqrt{t}}$$

if $t=0$ (sell now) then $V=K$.

Each value of V corresponding to a specific point of 't' represents dollar sum receivable at a different date.

Let us assume that the interest rate on the continuous compounding basis is at the level of r.

$$A(t) = Ve^{-rt} = Ke^{\sqrt{t}}e^{-rt}$$

$$\boxed{A(t) = Ke^{\sqrt{t}-rt}}$$

present value of V.

A possible date to sell (t) that maximizes A.

present value of r.

Our problem is to find 't' that maximises A.

$$\ln A(t) = \ln K + \ln e^{(\sqrt{t} - rt)}$$

$$\ln A(t) = \ln K + t^{1/2} - rt$$

differentiating both sides,

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{2}t^{-1/2} - r$$

$$\frac{dA}{dt} = A\left(\frac{1}{2}t^{-1/2} - r\right)$$

$\hookrightarrow = 0$ F.O.C

$$A \neq 0 \quad \therefore \quad \frac{1}{2}t^{-1/2} - r = 0$$

$$\frac{1}{2}t^{-1/2} = r$$

$$r = \frac{1}{2\sqrt{t}}$$

$$\frac{1}{2r} = \sqrt{t}$$

$$\boxed{t^* = \frac{1}{4r^2}}$$

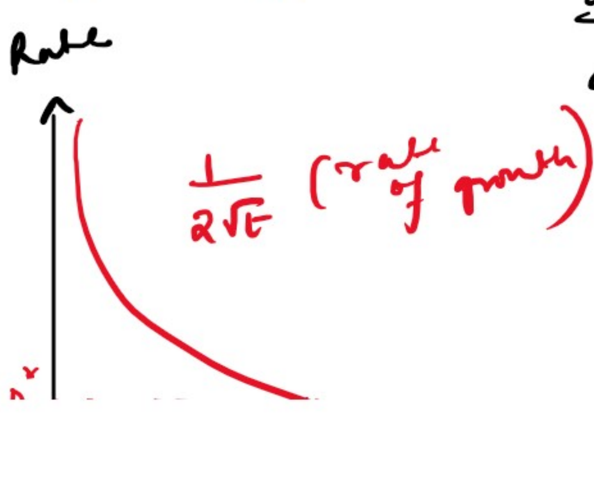
Say $r = 0.10$ how many years should store the wine?

$$t^* = \frac{1}{4(0.1)^2} = \underline{25 \text{ years}}$$

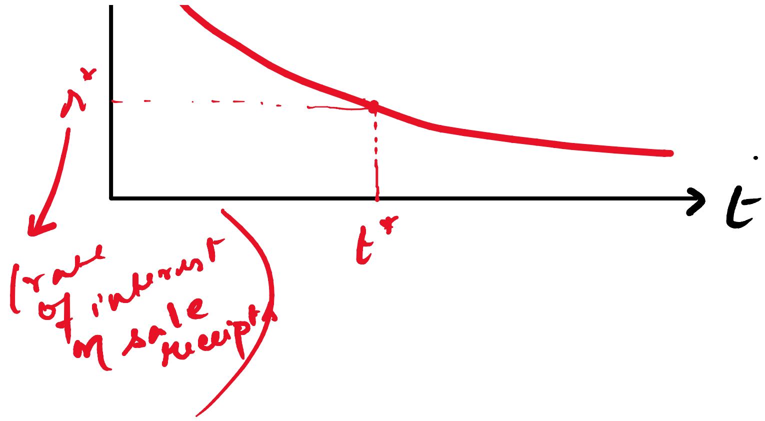
$$\frac{d^2A}{dt^2} = \frac{d}{dt}A\left(\frac{1}{2}t^{-1/2} - r\right)$$

$$= A\left(-\frac{1}{4}t^{-3/2}\right)$$

$$= -\frac{A}{4\sqrt{t^3}} < 0.$$



$$\frac{\cdots}{4\sqrt{t^3}} < 0 .$$



$e^{\sqrt{t}}$

Application : **A problem of Timber Cutting :**

$$V = 2^{\sqrt{t}}$$

Assuming a discount rate of r (on continuous compounding basis) also assuming cost during timber growth is 0.

What is the optimal time to cut timber for sale?

$$A(t) = V e^{-rt} = 2^{\sqrt{t}} e^{-rt}$$

$$\ln A = \ln 2^{\sqrt{t}} + \ln e^{-rt}$$

$$= t^{1/2} \ln 2 - rt$$

To maximise A, we must set $\frac{dA}{dt} = 0$

$$\frac{1}{A}\frac{dA}{dt} = \frac{1}{2} t^{-1/2} \ln 2 - r$$

$$\frac{dA}{dt} = A\left(\frac{\ln 2}{2\sqrt{t}} - r\right)$$

$A \neq 0$, $\quad \frac{dA}{\quad} = 0$

$$\frac{\ln 2}{2\sqrt{t}} - r = 0$$

$$\frac{\ln 2}{2\sqrt{t}} = r$$

$$\boxed{t^* = \left(\frac{\ln 2}{2r}\right)^2}$$

Q. If the value of wine grows according to the function $V = Ke^{2\sqrt{t}}$, how long should the dealer store the wine?

$$A = Ke^{2\sqrt{t} - rt}$$

$$\ln A = \ln K + 2\sqrt{t} - rt$$

$$\frac{1}{A}\frac{dA}{dt} = t^{-1/2} - r$$

$$\frac{dA}{dt} = A\left(t^{-1/2} - r\right)$$

$$t^{-1/2} - r = 0$$

$$t^{-1/2} = r$$

$$\boxed{t^* = \frac{1}{r^2}}$$

Envelope theorem ⇒ (Maximum value function).

The envelope theorem says that only the ... of change in an exogenous

The envelope theorem says that only the direct effect of change in an exogenous variable need to be considered, even though the exogenous variable may also enter the maximum-value-function indirectly as a part of the solution to the endogeneous choice variables.

In case of unconstrained optimisation

$$\max V = f(x, y, \phi)$$

$$f_x = f_y = 0$$

optimise and get $x^* = x^*(\phi)$

$$y^* = y^*(\phi)$$

S.O.C

f_{xx} f_{yy}

f_{xy}

$f_{xx} \cdot f_{yy} > f_{xy}^2$

saddle point $< f_{xy}^2$

$=$ inflection

$P_x \; P_y \; M$

$x^* = f(P_x, P_y, M)$

$y^* = f(P_x, P_y, M)$

Substitute in $V = f(x, y, \phi)$

$$\boxed{\max V} = V = f(x^*(\phi), y^*(\phi), \phi)$$

↳ max-value function.

↳ indirect-utility fn.

If we differentiate V w.r.t ϕ

$$\frac{dV}{d\phi} = f_x \frac{\partial x^*}{\partial \phi} + f_y \frac{\partial y^*}{\partial \phi} + f_\phi$$

$$\frac{dv}{d\phi} = f_\phi$$

Is envelope theorem

Envelope theorem for Constrained Optimisation.

$$\max \quad U = f(x, y; \phi)$$

$$\text{s.t.} \quad g(x, y; \phi) = 0$$

$$Z = f(x, y; \phi) + \lambda[0 - g(x, y; \phi)]$$

$$Z_x = f_x - \lambda g_x = 0 \quad \Rightarrow x^* = x(\phi)$$

$$Z_y = f_y - \lambda g_y = 0 \quad \Rightarrow y^* = y(\phi)$$

$$Z_\lambda = -g(x, y, \phi) = 0 \quad \Rightarrow \lambda = \lambda^*(\phi)$$

$$U^* = f(x^*(\phi), y^*(\phi), \phi) = V(\phi)$$

$$\frac{dV}{d\phi} = f_x \frac{\partial x^*}{\partial \phi} + f_y \frac{\partial y^*}{\partial \phi} + f_\phi \quad \text{--- (1)}$$

$$g(x^*(\phi), y^*(\phi), \phi) = 0$$

$$g_x \frac{\partial x^*}{\partial \phi} + g_y \frac{\partial y^*}{\partial \phi} + g_\phi = 0 \quad \text{--- (2)}$$

$$\frac{dV}{d\phi} = (f_x - \lambda g_x)\frac{\partial x^*}{\partial \phi} + (f_y - \lambda g_y)\frac{\partial y^*}{\partial \phi} + f_\phi - \lambda g_\phi \quad = Z_\phi$$

① $\boxed{\dfrac{dV}{d\phi} = Z_\phi}$ ✓

② $\boxed{V_{\phi\phi} \geq f_{\phi\phi}}$ ✓

$V = (x, y)$
$\hookrightarrow x^*, y^*$

Roy's Identity ⇒ One application of envelope theorem is derivation of Roy's Identity.

$V = f(P_x, P_y, M)$

Roy's identity is

$$\frac{\partial V / \partial P_x}{\partial V / \partial M} = -x$$

↑ optimum value of x & Marshallian dd fn of x.

$$\frac{\partial V / \partial P_y}{\partial V / \partial M} = -y$$

$$\boxed{V = \frac{m^2}{4 P_x P_y}}$$

$x, y.$

$\partial V \quad = \frac{2M}{\ldots}$

$$\frac{\partial v}{\partial p_x} = \frac{m^2}{4p_y}\left(-\frac{1}{p_x^2}\right)$$

$$\frac{\partial v}{\partial m} = \frac{2m}{4p_x p_y}$$

$$x = -\frac{\partial v/\partial p_x}{\partial v/\partial m} = \left[\frac{\frac{m^2}{4p_y p_x^2}}{\frac{2m}{4p_x p_y}}\right]$$

$$\boxed{x = \frac{m}{2p_x}}$$