## Alternating Series

Underlying seq $\{u_n\}$ $\Rightarrow$ series $= u_1 + u_2 + \cdots\cdots = \sum_{n=1}^{\infty} u_n$

$\therefore$ Construct the series as $u_1 - u_2 + u_3 - u_4 \cdots\cdots \Rightarrow$ Alternating series

## Leibnitz Test for Alternating Series:

An alternating series $u_1 - u_2 + u_3 - \cdots\cdots$ converges if the underlying seq $\{u_n\}$ monotonically decreases to zero.

Q. $\overset{u_1}{2} - \overset{u_2}{\frac{3}{2}} + \overset{u_3}{\frac{4}{3}} - \overset{u_4}{\frac{5}{4}} + \cdots\cdots$

$$u_n = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

Monotonically decreasing: $u_{n+1} - u_n = \frac{n+2}{n+1} - \frac{n+1}{n}$.

$$= \frac{\cancel{n^2} + \cancel{2n} - \cancel{n^2} - \cancel{2n} - 1}{n(n+1)} < 0$$

$$u_n = \frac{n+1}{n} \Rightarrow \lim_{n\to\infty} u_n = 1 \neq 0. \Rightarrow \text{Not convergent}.$$

Q. $\overset{u_1}{\frac{3}{1\cdot 2}} - \overset{u_2}{\frac{5}{2\cdot 3}} + \overset{u_3}{\frac{7}{3\cdot 4}} - \cdots\cdots\cdots$

$$u_n = \frac{2n+1}{n(n+1)}$$

$$u_{n+1} - u_n = \frac{2(n+1)+1}{(n+1)(n+2)} - \frac{2n+1}{n(n+1)} < 0.$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2n+1}{n(n+1)} = \lim_{n\to\infty} \frac{\frac{2}{n} + \frac{1}{n^2}}{1 + \frac{1}{n}} = \frac{0}{1} = 0.$$

convergent.

Convergence of series $\sum_{n=1}^{\infty} u_n \Leftrightarrow$ Existence of a sum (s)

Define: $\{S_n\}$ as the seq of partial sum of the given series

$$\{S_n\} = \{S_1, S_2, S_3, \ldots\ldots\}.$$

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3.$$

If $\{S_n\}$ is convergent $\Leftrightarrow \sum u_n$ is convergent

Note: Construct $\{S_n\}$ from $\sum u_n$ and check $\lim\limits_{n\to\infty} S_n$

Q. $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. Check for convergence and find the sum if the series is convergent.

$$u_n = \frac{1}{n(n+1)(n+2)}$$

$$S_n = u_1 + u_2 + \cdots + u_n.$$
$$= \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots\cdots + \frac{1}{n(n+1)(n+2)}$$

Compute: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left[\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)}\right]$

Note: Partial Fractions:

$$u_n = \frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \quad [\text{Find } A, B, C]$$

$$\frac{1}{n\cancel{(n+1)}(n+2)} = \frac{A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)}{n\cancel{(n+1)}(n+2)}$$

$n\cancel{(n+1)}(n+2)$

$n\cancel{(n+1)}(n+2)$

$$1 = A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)$$

Put $n=0 \Rightarrow 1 = A(1)(2) \Rightarrow A = 1/2$

Put $n=-1 \Rightarrow 1 = B(-1)(1) \Rightarrow B = -1$.

Put $n=-2 \Rightarrow 1 = C(-2)(-1) \Rightarrow C = 1/2$.

$$u_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\cdot\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}\cdot\frac{1}{n+2}$$

$$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$u_1 = \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) - \frac{1}{2}\left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$u_2 = \frac{1}{2}\left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) - \frac{1}{2}\left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

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$$u_n = \frac{1}{2}\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

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$$S_n = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$\lim_{n\to\infty} S_n = \frac{1}{4} \Rightarrow \sum u_n$ is convergent and converges to $(1/4)$.

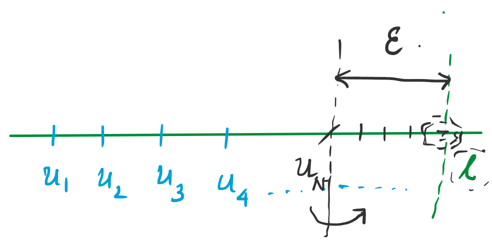
HW.

Q. $\sum u_n = \sum \frac{1}{(2n-1)(2n+1)}$. Check for convergence also find the sum if convergent.

## Cauchy sequence:

Convergent seq $\{u_n\} \to l$.

$$|u_n - l| < \varepsilon \;\forall\; n > N.$$



$|u_n - l| < \varepsilon \ \forall \ n > N.$

$u_1 \ \ u_2 \ \ u_3 \ \ u_4 \cdots u_N \cdots l$

**Cauchy seq:** A seq $\{u_n\}$ is said to be Cauchy seq if for any $\varepsilon > 0$ $\exists$ a natural no. $N$ s.t

$$|u_{n+p} - u_n| < \varepsilon \ \forall \ n > N.$$

If Cauchy seq $\Leftrightarrow$ convergent seq

**Cauchy criteria for convergence of series:**

For a series $\sum_{n=1}^{\infty} u_n$ to be convergent, then for any $\varepsilon > 0$ $\exists$ a natural no. $N$ s.t $|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon \ \forall n > N$

$\Rightarrow \ |S_{n+p} - S_n| < \varepsilon \ \forall \ n > N.$