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The cancellation of the Wimbledon tournament has led to a world surplus of tennis balls, and Santa has decided to use them as stocking fillers. He comes down the chimney with n identical tennis balls, and he finds k named stockings waiting for him.

Let $g(n, k)$ be the number of ways that Santa can put the n balls into the k stockings; for example, $g(2, 2) = 3$, because with two balls and two children, Miriam and Adam, he can give both balls to Miriam, or both to Adam, or he can give them one ball each.

(i) What is the value of $g(1, k)$ for $k \geqslant 1$?

(ii) What is the value of $g(n, 1)$?

(iii) If there are $n \geqslant 2$ balls and $k \geqslant 2$ children, then Santa can either give the first ball to the first child, then distribute the remaining balls among all k children, or he can give the first child none, and distribute all the balls among the remaining children. Use this observation to formulate an equation relating the value of $g(n, k)$ to other values taken by g.

(iv) What is the value of $g(7, 5)$?

(v) After the first house, Rudolf reminds Santa that he ought to give at least one ball to each child. Let $h(n, k)$ be the number of ways of distributing the balls according to this restriction. What is the value of $h(7, 5)$?

Alice, Bob and Charlie are well-known expert logicians.

(i) The King places a hat on each of their heads. Each of the logicians can see the others' hats, but not his or her own.

The King says "Each of your hats is either black or white, but you don't all have the same colour hat".

All four are honest, and all trust one another.

The King now asks Alice "Do you know what colour your hat is?".

Alice says "Yes, it's white".

What colour are the others' hats? [Hint: think about how Alice can deduce that her hat is white.]

(ii) The King now changes some of the hats, and again says "Each of your hats is either black or white, but you don't all have the same colour hat". He now asks Alice "Do you know what colour your hat is?".

Alice replies "No". ⇒ A sees 1 Black & 1 white.

B deduce that he and C have opp colours

Can Bob work out what colour his hat is? Explain your answer. [Hint: what can Bob deduce from the fact that Alice can't tell what colour her hat is?]

(iii) The King now changes some of the hats, and then says "Each of your hats is either black or white. At least one of you has a white hat." $N \geq 1 \Rightarrow W = 1, 2, 3$

He now asks them all "Do you know what colour your hat is?". They all simultaneously reply "No".

2W or 3W

What can you deduce about the colour of their hats? Explain your answer.

(iv) He again asks "Do you know what colour your hat is?" Alice says "No", but Bob and Charlie both say "Yes" (all three answer simultaneously).

What colour are their hats? Explain your answer.

B & C → W

A → ?

(i) Alice, Bob, Charlie and Dianne each make the following statements:

Binary logic

Alice: I am telling the truth. T. L.

Binary logic

(i) Alice, Bob, Charlie and Dianne each make the following statements:

Alice:	I am telling the truth. T L
Bob:	Alice is telling the truth. T L
Charlie:	Bob is telling the truth. L
Dianne:	Charlie is lying. T

Only one of the 4 people is telling the truth. Which one? Explain your answer.

(ii) They now make the following statements:

Alice:	Bob is lying. T F
Bob:	Charlie is lying. F T
Charlie:	I like beer. T F
Dianne:	2+2=4. T T

Now two of the four people are telling the truth. Which two? Explain your answer.

(iii) They are now joined by Egbert. They each make the following statements:

Alice:	I like wine. F
Bob:	Charlie is lying. F
Charlie:	Alice is lying. T
Dianne:	Alice likes beer. T
Egbert:	Alice likes beer. T

Now three of the five people are telling the truth. Which ones? Explain your answer.

A flexadecimal number consists of a sequence of digits, with the rule that the rightmost digit must be 0 or 1, the digit to the left of it is 0, 1, or 2, the third digit (counting from the right) must be at most 3, and so on. As usual, we may omit leading digits if they are zero. We write flexadecimal numbers in angle brackets to distinguish them from ordinary, decimal numbers. Thus $\langle 34101\rangle$ is a flexadecimal number, but $\langle 231\rangle$ is not, because the digit 3 is too big for its place. (If flexadecimal numbers get very long, we will need 'digits' with a value more than 9.)

The number 1 is represented by $\langle 1\rangle$ in flexadecimal. To add 1 to a flexadecimal number, work from right to left. If the rightmost digit d_1 is 0, replace it by 1 and finish. Otherwise, replace d_1 by 0 and examine the digit d_2 to its left, appending a zero at the left if needed at any stage. If $d_2 < 2$, then increase it by 1 and finish, but if $d_2 = 2$, then replace it by 0, and again move to the left. The process stops when it reaches a digit that can be increased without becoming too large. Thus, the numbers 1 to 4 are represented as $\langle 1\rangle$, $\langle 10\rangle$, $\langle 11\rangle$, $\langle 20\rangle$.

$1 = \langle 01\rangle$ $2^2\ 2^1\ 2^0$

$2 = \langle 10\rangle$ $2\times 1 + 0$

$3 = \langle 11\rangle$ 0-2

$5! = 120$

Binary

1 = 1
10 = 2
11 = 3
100 = 4
101 = 5
110 = 6

0,1

$2^1 + 0$

$2^2 + 0 + 0$

$6 = \langle 100\rangle$

$24 = \langle 1000\rangle$

(i) Write the numbers from 5 to 13 in flexadecimal.

(ii) Describe a workable procedure for converting flexadecimal numbers to decimal, and explain why it works. Demonstrate your procedure by converting $\langle 1221\rangle$ to decimal.

(iii) Describe a workable procedure for converting decimal numbers to flexadecimal, and demonstrate it by converting 255 to flexadecimal.

$255 = 2\times 5! + 15$

$= 2\times 5! + 0\times 4! + 2\times 3! + 2\times 2! + 1$

$\langle 20221\rangle$

(iv) We could add flexadecimal numbers by converting them to decimal, adding the decimal numbers and converting the result back again. Describe instead a procedure for addition that works directly on the digits of two flexadecimal numbers, and demonstrate it by performing the addition $\langle 1221\rangle + \langle 201\rangle$.

(v) Given a flexadecimal number, how could you test whether it is a multiple of 3 without converting it to decimal?

(vi) If the $\langle 100000\rangle$ arrangements of the letters *abcdef* are listed in alphabetical order and numbered $\langle 0\rangle$: *abcdef*, $\langle 1\rangle$: *abcdfe*, $\langle 10\rangle$: *abcedf*, etc., what arrangement appears in position $\langle 34101\rangle$ in the list?

+91 8105735888

$12345 = 1\times 10^4 + 2\times 10^3 + 3\times 10^2 + 4\times 10^1 + 5$