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Constrained Optimisation

(Lagrange Method).

1. Objective fn.
2. Constraint fn.

Objective is to maximise $U = U(x, y)$

sbj to $M = P_x \cdot x + P_y \cdot y$ constraint

$M - P_x \cdot x - P_y \cdot y = 0$

income. Price of commodity

Lagrangian expression can be written as,

$$\boxed{L = U(x, y) + \lambda (M - P_x \cdot x - P_y \cdot y)}$$

F.O.Cs

$$\frac{\partial L}{\partial x} = 0 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial y} = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad \text{--- (3)}$$

x and y (optimum value)

$x^* = x(P_x, P_y, M)$

$y^* = y(P_x, P_y, M)$

demand fn of x and Y

ordinary / Marshallian.

S.O.C

$$|H| = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial \lambda} \\ \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial x} & \frac{\partial^2 L}{\partial \lambda \partial y} & \frac{\partial^2 L}{\partial \lambda^2} \end{vmatrix}$$

Bordered Hessian determinant

✓ For max, $|H| > 0$

✓ for min, $|H| < 0$

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Q1 optimise the utility function $U = 4xy - y^2$ subject to budget constraint $2x + y = 6$.

Our objective is to Max $U = 4xy - y^2$

subj to $2x + y = 6$

Lagrangian Expression can be written as,

$$L = (4xy - y^2) + \lambda(2x + y - 6)$$

0, 4, 2
4, -2, 1

F.O.C

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \boxed{4y + 2\lambda} = 0$$

$$\Rightarrow \lambda = -\frac{4y}{2} = -2y \quad \text{—— (1)}$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow \boxed{4x - 2y + \lambda} = 0$$

$$\Rightarrow \lambda = 2y - 4x \quad \text{—— (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \boxed{2x + y - 6} = 0$$

$$\Rightarrow 2x + y = 6 \quad \text{—— (3)}$$

Comparing (1) and (2), we get

$$-2y = 2y - 4x$$

$$-4y = -4x$$

$$\boxed{x = y} \quad \text{—— (4)}$$

Let us substitute $x = y$ in eq (3) (or the budget constraint).

$$2x + x = 6$$

$$3x = 6$$

$$\boxed{x = 2} \quad \therefore \boxed{y = 2}$$

The S.O.C for utility maximisation,

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$$|H| = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial \lambda} \\ \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial x} & \frac{\partial^2 L}{\partial \lambda \partial y} & \frac{\partial^2 L}{\partial \lambda^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 & 2 \\ 4 & -2 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 0 - 4(0-2) + 2(4+4)$$

$$= 8 + 16 = 24 > 0$$

$\therefore |H| > 0$ for $x^* = y^* = 2$.

$\therefore$ max $U = 4x^*y^* - y^{*2} = 4(2)(2) - (2)^2$
$= 16 - 4 = \underline{\underline{12}}$

Q2

Max $U = x^\alpha y^\beta$
subj to $M = P_x \cdot x + P_y \cdot y$

} Find the ordinary demand fn of x and y.

$$L = (x^\alpha y^\beta) + \lambda (M - P_x \cdot x - P_y \cdot y)$$

f.o.c $\frac{\partial L}{\partial x} = 0 \Rightarrow \alpha x^{\alpha-1} y^\beta - \lambda P_x = 0$

$$\Rightarrow \lambda = \frac{\alpha x^{\alpha-1} y^\beta}{P_x} \quad \text{--- (1)}$$

∂x

$$\Rightarrow \lambda = \frac{\alpha x^{\alpha-1} y^{\beta}}{P_x} \quad - (1)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x^{\alpha} \cdot \beta y^{\beta-1} - \lambda P_y = 0$$

$$\Rightarrow \lambda = \frac{\beta x^{\alpha} y^{\beta-1}}{P_y} \quad - (2)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow M - P_x \cdot x - P_y \cdot y = 0$$

$$\Rightarrow M = x \cdot P_x + P_y \cdot y \quad — (3)$$

comparing (1) and (2),

$$\frac{\alpha x^{\alpha-1} y^{\beta}}{P_x} = \frac{\beta x^{\alpha} y^{\beta-1}}{P_y}$$

$$\frac{P_y \, \alpha \, x^{\alpha-1}}{\beta \, P_x x^{\alpha}} = \frac{y^{\beta-1}}{y^{\beta}}$$

$$\frac{\alpha}{\beta} \frac{P_y}{P_x} \frac{1}{x} = \frac{1}{y}$$

$$\Rightarrow \boxed{x = \frac{\alpha P_y}{\beta P_x} \cdot y} \quad — (4)$$

Substitute (4) in Budget Constraint,

$$M = x P_x + y P_y$$

$$M = \frac{\alpha}{\beta} \frac{P_y}{\cancel{P_x}} y \cancel{P_x} + y P_y$$

$$M = y P_y \left(\frac{\alpha}{\beta} + 1 \right)$$

$$M = y P_y \left(\frac{\alpha}{\beta} + 1 \right)$$

$$M = y P_y \left(\frac{\alpha + \beta}{\beta} \right)$$

$$\therefore \boxed{y^* = \frac{M \cdot \beta}{P_y \cdot (\alpha + \beta)}}$$

$$\therefore x^* = \frac{\alpha}{\beta} \cdot \frac{P_y}{P_x} \cdot y^*$$

$$= \frac{\alpha}{\cancel{\beta}} \cdot \frac{\cancel{P_y}}{P_x} \cdot \frac{M}{\cancel{P_y}} \cdot \frac{\cancel{\beta}}{(\alpha + \beta)}$$

$$\boxed{x^* = \frac{M \cdot \alpha}{P_x (\alpha + \beta)}}$$

$$\boxed{\begin{array}{l} U = x^{\alpha} y^{\beta} \\ \therefore \text{Indirect utility fn} \\ V = \left(\frac{M}{P_x} \cdot \frac{\alpha}{\alpha + \beta} \right)^{\alpha} \left(\frac{M}{P_y} \cdot \frac{\beta}{\alpha + \beta} \right)^{\beta} \end{array}}$$

Indirect utility function.

$U = U(x, y)$ after optimisation we find $x^* = x(P_x, P_y, M)$

$y^* = y(P_x, P_y, M)$

$= U\left[x^*(P_x, P_y, M), y^*(P_x, P_y, M) \right]$

$= V(P_x, P_y, M) \approx V$ (indirect utility fn.)

Roy's Identity [derivation of x^* and y^* (ordinary demand fn from indirect utility function (v)]

Let $V = V(P_x, P_y, M)$

then, $x^* = -\dfrac{\partial V/\partial P_x}{\partial V/\partial M}$

$y^* = -\dfrac{\partial V/\partial P_y}{\partial V/\partial M}$

derivation of compensated demand function:

Obj: Minimise $P_x \cdot x + P_y \cdot y$

subj to: $\overline{U} = U(x, y)$

$$L = (P_x \cdot x + P_y \cdot y) + \lambda(\overline{U} - U(x, y))$$

F.O.C $\partial L/\partial x = 0$ $\quad \dfrac{\partial L}{\partial y} = 0 \quad \dfrac{\partial L}{\partial \lambda} = 0$

we will calculate x and y

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Example Let Utility fn be $U = x \cdot y$
subject to $M = P_x \cdot x + P_y \cdot y$.

Solve: (1) Ordinary dd fn of x and y.
(2) Indirect utility fn.
(3) verify Roy's Identity
(4) Derive compensated dd fn of x and y.

Solution: (1) Max $U = xy$
subj to $M = P_x \cdot x + P_y \cdot y$

$$L = xy + \lambda(M - P_x \cdot x - P_y \cdot y)$$

F.O.C: $\frac{\partial L}{\partial x} = 0 \Rightarrow y - \lambda P_x = 0$

$$\lambda = \frac{P_x}{y} \quad \text{— (1)}$$

$\frac{\partial L}{\partial y} = 0 \Rightarrow x - \lambda P_y = 0$

$$\lambda = \frac{P_y}{x} \quad \text{— (2)}$$

$\partial L / \partial \lambda = 0 \Rightarrow M = xP_x + yP_y$ — (3)

Comparing (1) and (2)

$$\frac{P_y}{x} = \frac{P_x}{y} \Rightarrow \boxed{x = \frac{P_y}{P_x} \cdot y} \quad \text{— (4)}$$

Substituting 'x' in (3)

$$M = \frac{P_y}{\cancel{P_x}} y \cdot \cancel{P_x} + y P_y$$

or, $M = 2yP_y$

or, $\boxed{y = \frac{M}{2P_y}}$ — (5)

(4) M

from (4), $\boxed{x = \frac{M}{2P_x}}$ — (6)

(2) Indirect utility fn, $U = xy$

$$V = \left[\frac{M}{2P_y}\right]\left[\frac{M}{2P_x}\right]$$

$$\boxed{V = \frac{m^2}{4P_y P_x}}$$

(3) Indirect utility fn

$$V = \frac{m^2}{4P_y P_x}$$

$$\frac{\partial v}{\partial P_x} = \frac{m^2}{4P_y(-P_x^2)} = -\frac{m^2}{4P_y P_x^2}$$

$$\partial v/\partial P_y = -\frac{m^2}{4P_y^2 P_x}$$

$$\frac{\partial v}{\partial m} = \frac{2m}{4P_y P_x} = \frac{m}{2P_y P_x}$$

Using Roy's Identity:

$$x^* = -\frac{\partial v/\partial P_x}{\partial v/\partial m} = -\left[\frac{M^{\cancel{2}}}{\cancel{4}_2 \cancel{P_y} P_x^{\cancel{2}}} \times \frac{\cancel{2}\cancel{P_y}\cancel{P_x}}{\cancel{M}}\right]$$

$$\boxed{x^* = \frac{M}{2P_x}}$$

$$y^* = -\frac{\partial v/\partial P_y}{\partial v/\partial m} = -\left[-\frac{m^{\cancel{2}}}{\cancel{4}_2 P_y^{\cancel{2}} \cancel{P_x}} \cdot \frac{\cancel{2}\cancel{P_x}\cancel{P_y}}{\cancel{M}}\right]$$

$$\partial v / \partial M$$

$$\boxed{y^* = \frac{M}{2P_y}}$$

(4) Objective is to Minimise $P_x \cdot x + P_y \cdot y$

sbj to, $\bar{U} = xy$

$$L = (P_x \cdot x + P_y \cdot y) + \lambda^h (\bar{U} - xy)$$

FOC $\frac{\partial L}{\partial x} = 0 \Rightarrow P_x - \lambda^h y = 0 \Rightarrow \lambda^h = P_x / y$ — (1)

$\partial L / \partial y = 0 \Rightarrow P_y - \lambda^h x = 0 \Rightarrow \lambda^h = P_y / x$ — (2)

$\partial L / \partial M = 0 \Rightarrow \bar{U} - xy = 0$

$\Rightarrow \bar{U} = xy$ — (3) ✓

Comparing (1) and (2): $\frac{P_x}{y} = \frac{P_y}{x}$

or, $\boxed{x = \frac{P_y}{P_x} \cdot y}$ — (4)

Putting (4) in (3),

$\bar{U} = xy \Rightarrow \bar{U} = \frac{P_y}{P_x} \cdot y \cdot y$

$\Rightarrow y^2 = \frac{P_x \cdot \bar{U}}{P_y}$

$$\Rightarrow \boxed{y = \sqrt{\frac{P_x \cdot \bar{U}}{P_y}}}$$

from (4), $x = \frac{P_y}{P_x} \cdot y = \frac{P_y}{P_x} \cdot \sqrt{\frac{P_x \cdot \bar{U}}{P_y}}$

$= \frac{\sqrt{P_y}}{\sqrt{P_x}} \cdot \sqrt{\bar{U}}$

Compensated or Hicksian demand function

$$\boxed{x = \sqrt{\frac{P_y \cdot \bar{U}}{P_x}}}$$

Homework

(1) $U = (x+2)(y+1)$
Subj to: $4x + 6y = 130$
Find optimum value of x and y.

(2) $U = (x_1 - a_1)^{\alpha} (x_2 - a_2)^{1-\alpha}$
Subj to: $M = xP_x + y \cdot P_y$
Find the ordinary demand function for x and y.

(3) Construct ordinary and compensated demand function for the given utility function $u = 2x_1x_2 + x_2$ and budget equation $M = P_1x_1 + P_2x_2$.

(4) Roy's Identity.