Digression: Distributions derived from the Normal Distribution.

r.s: $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

x_i's : sampling units.

$x_i \sim N(\mu, \sigma^2) \Rightarrow$ all the sampling units have the same (identical) distributions and are independent

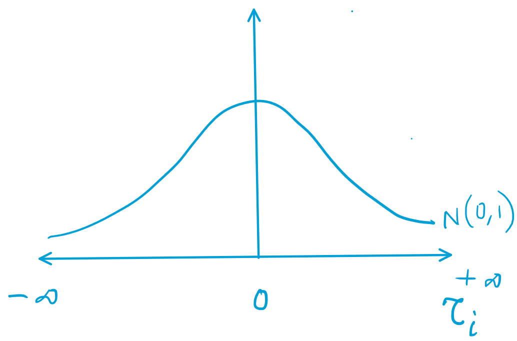
$\left.\begin{aligned} &\Rightarrow E(x_i) = \mu \\ &\Rightarrow Var(x_i) = \sigma^2 \end{aligned}\right\} \forall i$.

(i) Standard Normal Distribution:-

$x_i \sim N(\mu, \sigma^2)$

Define a r.v $\tau_i = \underbrace{\left(\frac{x_i - \mu}{\sigma}\right)}_{\text{constructed r.v.}} \sim N(0,1)$

Symmetric about zero.



(ii) Chi-square Distribution:-

$$Y = \underbrace{\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2}_{\text{constructed r.v.}} \sim \chi^2_{(n)}.$$

Chi-sq distribution with $df = n$.

[df = Total no. of independent variables

or Total no. of variables − Total no. of restrictions]

$$Y = \sum_{i=1}^{n} \tau_i^2 = \sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2 = \underbrace{\left(\frac{x_1 - \mu}{\sigma}\right)^2 + \left(\frac{x_2 - \mu}{\sigma}\right)^2 + \dots + \left(\frac{x_n - \mu}{\sigma}\right)^2}_{\text{independent}}.$$

[summation has 'n' independent terms, $\therefore df = n$].

Let $Y' = \sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma}\right)^2$, where $\bar{x}$ is the sample mean.

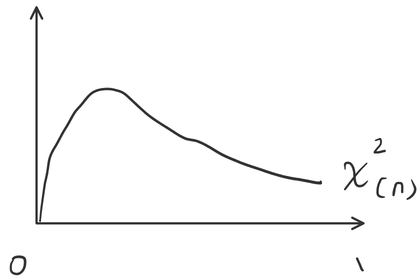
Then $Y' \sim \chi^2_{(n-1)}$,

$$Y' = \sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma}\right)^2 = \underbrace{\left(\frac{x_1 - \bar{x}}{\sigma}\right)^2 + \left(\frac{x_2 - \bar{x}}{\sigma}\right)^2 + \cdots + \left(\frac{x_n - \bar{x}}{\sigma}\right)^2}$$

No. of terms = n.

Restriction $\Rightarrow \sum(x_i - \bar{x}) = 0$.

$\therefore$ df = $(n-1)$.



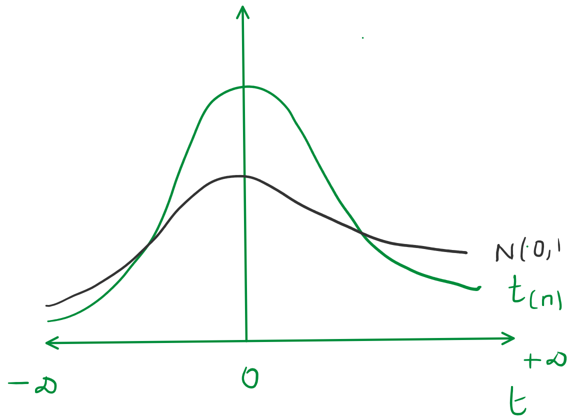
chi-sq distribution is positively skewed.

& defined over all positive real values.

(iii) t-distribution :-

$$t = \frac{\tau}{\sqrt{\chi^2_{(n)}/n}} \sim t_{(n)}$$

$\rightarrow$ constructed r.v.



Symmetric about zero.

t-distn is leptokurtic (more steeper)

τ is mesokurtic.

(iv) F-distribution :-

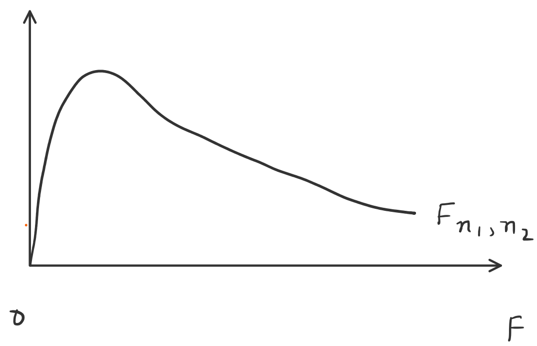
$\chi^2_{(n_1)}$ $\chi^2_{(n_2)}$

Consider 2 independent chi-sq variates, $\chi^2_{(1)}$ and $\chi^2_{(2)}$ with n_1 and n_2 df respectively.

$$F = \frac{\chi^2_{(1)}/n_1}{\chi^2_{(2)}/n_2} \sim F_{n_1, n_2}.$$

$\rightarrow$ constructed r.v.

F-distribution with df = (n_1, n_2)



F-distribution is positively skewed, & defined for all +ve real values.

Note: $F_{1,n} = \dfrac{\chi^2_{(1)}/1}{\chi^2_{(2)}/n} = \dfrac{\chi^2_{(1)}}{\chi^2_{(2)}/n} = \dfrac{\tau^2}{\chi^2_{(n)}/n}$

$$\sqrt{F_{1,n}} = \frac{\tau}{\sqrt{\chi^2_{(n)}/n}} = t_{(n)}.$$

Continuation:

$$\frac{(\hat{\beta} - \beta)^2}{\sigma^2/\Sigma(x_i - \bar{x})^2} \sim \chi^2_{(1)}$$

$\rightarrow$ independent

Result: $\dfrac{\Sigma e_i^2}{\sigma^2} \sim \chi^2_{(n-2)}$

$$F = \frac{\dfrac{(\hat{\beta} - \beta)^2}{\cancel{\sigma^2}/\Sigma(x_i - \bar{x})^2}}{\dfrac{\Sigma e_i^2}{\cancel{\sigma^2}}\Big/(n-2)} \sim F_{1,(n-2)}$$

$$= \frac{(\hat{\beta} - \beta)^2 / \Sigma(x_i - \bar{x})^2}{\left[\Sigma e_i^2/(n-2)\right]}$$

$$= \frac{(\hat{\beta} - \beta)^2 / \Sigma(x_i - \bar{x})^2}{\hat{\sigma}^2}.$$

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HW: $\sqrt{F} = ?$

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