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Find : $\int \frac{dx}{\sqrt{4x-x^2}}$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$ax^2+bx+c \to (x+k)^2 \pm m^2$.

$(x+k)^2+m^2 \Rightarrow (x+k) = m\tan\theta$.

$(x+k)^2-m^2 \Rightarrow (x+k) = m\sec\theta$.

$\boxed{m^2-(x+k)^2 \Rightarrow (x+k) = m\sin\theta}$

$4x-x^2 = 4+4x-x^2-4$

$= 4-(x^2-4x+4) = \boxed{2^2-(x-2)^2}$

$\boxed{x-2 = 2\sin\theta}$ $\quad \theta = \sin^{-1}\left(\frac{x-2}{2}\right)$

$dx = 2\cos\theta\, d\theta$.

$4-4\sin^2\theta$

$= 4(1-\sin^2\theta) = 4\cos^2\theta$.

$$I = \int \frac{2\cos\theta\, d\theta}{\sqrt{4\cos^2\theta}} = \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta = \theta + c = \sin^{-1}\left(\frac{x-2}{2}\right) + c.$$

$\int \sqrt{\tan x}\, dx$

$\tan x = y^2 \to \sec^2 x\, dx = 2y\, dy$.

$dx = \frac{2y\,dy}{\sec^2 x} = \frac{2y\,dy}{1+\tan^2 x} = \frac{2y\,dy}{1+y^4}$.

$$I = \int y \cdot \frac{2y\,dy}{1+y^4} = \int \frac{2y^2 dy}{1+y^4} = \int \left(\frac{y^2+1+y^2-1}{1+y^4}\right) dy = \underbrace{\int\left(\frac{y^2+1}{y^4+1}\right)dy}_{I_1} + \underbrace{\int\left(\frac{y^2-1}{1+y^4}\right)dy}_{I_2}.$$

$$I_1 = \int \frac{y^2+1}{1+y^4}\, dy = \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\, dy.$$

$y-\frac{1}{y} = z$.

$\left(1+\frac{1}{y^2}\right)dy = dz$.

$$= \int \frac{\left(1+\frac{1}{y^2}\right)}{\left(y-\frac{1}{y}\right)^2+2}\, dy$$

$$= \int \frac{dz}{z^2+2}$$

$z = \sqrt{2}\tan\theta \qquad dz = \sqrt{2}\sec^2\theta\, d\theta$.

$$= \int \frac{\sqrt{2}\cancel{\sec^2\theta}\, d\theta}{2\cancel{\sec^2\theta}} = \frac{1}{\sqrt{2}}\theta = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{(y-1/y)}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan x-1}{\sqrt{2\tan x}}\right]$$

$$I_2 = \int \frac{y^2-1}{1+y^4}\, dy = \int \frac{1-1/y^2}{y^2+\frac{1}{y^2}}\, dy$$

$\boxed{y+\frac{1}{y} = \sqrt{2}\sec\theta} \to \left(1-\frac{1}{y^2}\right)dy = \sqrt{2}\sec\theta\tan\theta\, d\theta$.

$$= \int \frac{(1-1/y^2)\,dy}{\left(y+\frac{1}{y}\right)^2-2} = \int \frac{\sqrt{2}\sec\theta\,\cancel{\tan\theta}\, d\theta}{2\tan^{\cancel{2}}\theta} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\theta\, d\theta = -\frac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\theta+\cot\theta\right|$$

$\int \operatorname{cosec}\theta\, d\theta = -\log|\cot\theta+\operatorname{cosec}\theta|$

$= \log\left|\frac{1}{\cot\theta+\operatorname{cosec}\theta}\right|$

$\log|\cot\theta+\operatorname{cosec}\theta| = y. = -\int \operatorname{cosec}\theta\, d\theta$

$\frac{1}{\cot\theta+\operatorname{cosec}\theta}.(-\operatorname{cosec}^2\theta - \operatorname{cosec}\theta\cot\theta)\, d\theta = dy.$

$-\operatorname{cosec}\theta\, d\theta = dy$

$y = \int -\operatorname{cosec}\theta\, d\theta$

$\operatorname{cosec}\theta+\cot\theta = \frac{1+\cos\theta}{\sin\theta}$.

$\cos\theta = \frac{\sqrt{2}}{\ldots}$ $\quad \sin\theta = \sqrt{1-\frac{2}{(y+1/y)^2}} = \sqrt{\frac{y^2+1/y^2}{(y+1/y)^2}} = \frac{\sqrt{y^2+1/y^2}}{y+1/y}$.

$$\csc\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{y+\frac{1}{y}}{\sqrt{2}} \qquad \cos\theta = \frac{\sqrt{2}}{y+\frac{1}{y}} \qquad \sin\theta = \sqrt{1-\frac{2}{(y+1/y)^2}} = \sqrt{\frac{y^2+1/y^2}{(y+1/y)^2}} = \frac{\sqrt{y^2+1/y^2}}{y+1/y}$$

$$\csc\theta+\cot\theta = \frac{\frac{y+1/y+\sqrt{2}}{y+1/y}}{\frac{\sqrt{y^2+1/y^2}}{y+1/y}} = \frac{y+1/y+\sqrt{2}}{\sqrt{y^2+1/y^2}} = \frac{y^2+\sqrt{2}y+1}{\sqrt{y^4+1}} = \frac{\tan x+\sqrt{2\tan x}+1}{\sqrt{\sec^2 x}}$$

$$= \frac{\tan x+\sqrt{2\tan x}+1}{\sec x}$$

$$\int \sqrt[3]{\tan x}\, dx.$$

$$\tan x = y^3. \qquad \sec^2 x\, dx = 3y^2\, dy. \qquad dx = \frac{3y^2 dy}{1+y^6}.$$

$$I = \int \frac{y \cdot 3y^2 dy}{1+y^6} = \int \frac{3y^3 dy}{1+y^6}. \qquad 1+y^6 = (1+y^2)(1-y^2+y^4)$$

factors in the denominator $\Rightarrow$ partial fractions.

$$= 3\int \frac{y^3 dy}{(y^2+1)(y^4-y^2+1)} \qquad \frac{y^3}{(y^2+1)(y^4-y^2+1)} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}.$$

$$y^4-y^2+1 = \left[y^4 - 2\cdot\tfrac{1}{2}y^2+\left(\tfrac{1}{2}\right)^2\right] - \left(\tfrac{1}{2}\right)^2+1 = \left(y^2-\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2.$$

den $\to ax^2+bx+c$

num $\to Ax+B$.

$$y^3 = (Ay+B)(y^4-y^2+1) + (Cy+D)(y^2+1)$$

$y=0$: $B+D=0 \Rightarrow \boxed{B=-D}$ — ①

$y=1$: $(A+B)+(C+D)2 = 1$ $\quad\boxed{A-B+2C=1}$ — ②

$y=-1$: $(-A+B)+(-C+D)\times 2 = -1$

$-A-B-2C=-1$

$\boxed{A+B+2C=1}$ — ③

$\boxed{B=0=D}$ ✓ $\quad \boxed{A+2C=1}$ ④

$y=2$: $(2A+B)\times 13 + (2C+D)\times 5 = 8$.

$26A+10C = 8$.

$\boxed{13A+5C=4}$ ⑤

$13A+26C=13$

$13A+5C=4$

$21C=9$

✓ $\boxed{C=3/7}$

$$A = 1-2C = 1-\frac{6}{7} = \frac{1}{7} \qquad \boxed{A=\frac{1}{7}} ✓$$

$$I = 3\left[\int \frac{(\frac{1}{7})y}{y^2+1}dy + \int \frac{\frac{3}{7}y\ dy}{(y^2-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right] = 3\left[\frac{1}{7}\cdot\frac{1}{2}\log|y^2+1| + \frac{3}{7}\cdot\frac{\sqrt{3}}{4}\cdot\frac{1}{3/4}\tan^{-1}\left(\frac{2(y^2-1/2)}{\sqrt{3}}\right)\right]$$

$y^2+1=z$ $\quad$ $y\,dy = \frac{1}{2}dz$.

$2y\,dy = dz$

$y^2-1/2 = \frac{\sqrt{3}}{2}\tan\theta$

$2y\,dy = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$.

$y\,dy = \frac{\sqrt{3}}{4}\sec^2\theta\, d\theta$.

$$= \frac{3}{14}\left[\log|y^2+1| + 2\sqrt{3}\tan^{-1}\left(\frac{2y^2-1}{\sqrt{3}}\right)\right]$$

$$= \frac{3}{14}\left[\log\left|(\tan x)^{2/3}+1\right| + 2\sqrt{3}\tan^{-1}\left(\frac{2(\tan x)^{2/3}-1}{\sqrt{3}}\right)\right] + C$$

$$I = \int \sin^n x\, dx \qquad \int \cos^n x\, dx.$$

$$I = \int \sin^2 x\, dx \,/ \int \cos^2 x\, dx \qquad \cos 2x = 1-2\sin^2 x \to \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos 2x = 2\cos^2 x - 1 \to \cos^2 x = \frac{1+\cos 2x}{2}.$$

$$I = \int \sin^3 x\, dx \,/ \int \cos^3 x\, dx$$

$$d(\cos x) = -\sin x\, dx.$$

$$= \int \sin^2 x \cdot (\sin x\, dx)$$

$$= \int (1-\cos^2 x)(-d(\cos x)) = \int -d(\cos x) + \int \cos^2 x\, d(\cos x) = -\cos x + \frac{\cos^3 x}{3}.$$

$$\int \frac{\sin x}{\sin x + \cos x} dx$$

$$\int \frac{x-1}{(x+1)^3} e^x \, dx \qquad \int \frac{(x^2+1)e^x}{(x+1)^2} dx \qquad \int \frac{e^x(1+\sin x)dx}{1+\cos x}$$

$$\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}} dx$$