Algebra

$ax^2+bx+c = k(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow$ degree mismatch

(1) $\frac{p}{2x} = \frac{a}{x+c} + \frac{b}{x-c}$, roots are equal. find p.

$$\frac{p}{2x} = \frac{a(x-c)+b(x+c)}{x^2-c^2}$$

$$\Rightarrow (2a+2b-p)x^2 - 2c(a-b)x + pc^2 = 0$$

$B^2-4AC=0$ $\quad 4c^2(a-b)^2 - 4pc^2(2a+2b-p) = 0$

$$\Rightarrow (a-b)^2 - 2p(a+b) + p^2 = 0$$

$$\Rightarrow (a+b)^2 - 2p(a+b) + p^2 + (a-b)^2 = (a+b)^2$$

$$\Rightarrow \{(a+b)-p\}^2 = 4ab$$

$$\Rightarrow (a+b)-p = \pm 2\sqrt{ab}$$

$$\Rightarrow p = a \pm 2\sqrt{ab} + b = (\sqrt{a} \pm \sqrt{b})^2$$

solve this equation in terms of 'p' by quad. formula

(2) $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a) = 0$. Classify the roots.

$$3x^2 - 2(a+b+c)x + (ab+bc+ca) = 0$$

$$D = 4\left[(a+b+c)^2 - 3(ab+bc+ca)\right]$$

$$= 4\left[a^2+b^2+c^2-ab-bc-ca\right]$$

$$= 2\left[a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2\right]$$

$$= 2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

equal if $a=b=c$, else real & positive

$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$. Roots equal in magnitude, opposite in sign. Find $(p+q)$

$$\frac{2x+p+q}{(x+p)(x+q)} = \frac{1}{r}$$

$$x^2 + (p+q-2r)x + pq - (p+q)r = 0$$

Find sum of roots

Algebra

$x^2+2ax+b=0 \rightarrow$ roots α, β. Find equation with rational coefficients one of whose roots is $\dfrac{\alpha+\beta+\sqrt{\alpha^2+\beta^2}}{\alpha+\beta-\sqrt{\alpha^2+\beta^2}}$

$\alpha+\beta=-2a, \quad \alpha\beta=b$

Sum of roots $= 2(\alpha+\beta) = -4a$

Product $= (\alpha+\beta)^2-\alpha^2-\beta^2 = 2\alpha\beta = 2b$

$\therefore$ new eq^n $\Rightarrow$ $x^2+4ax+2b=0$

If $x^2-x-a=0$ has integer roots, find possible values of a

$a \in [6,100]$

$D = 1+4a = (2n+1)^2$

$\Rightarrow 1+4a = 4n^2+4n+1 \Rightarrow a = n(n+1)$

✓ $a = \{6,12,20,30,42,56,72,90\}$ $\quad \therefore$ 8 solutions

$*$ $\dfrac{(x-a)(x-b)}{(c-a)(c-b)} + \dfrac{(x-b)(x-c)}{(a-b)(a-c)} + \dfrac{(x-c)(x-a)}{(b-c)(b-a)} = 1$

$x^2+x+1=0$. Find equation whose roots are α^{19}, β^7

(use cube roots of unity)

$$\left.\begin{aligned} ax^2+bx+c&=0 \\ a'x^2+b'x+c'&=0 \end{aligned}\right\} \quad (a'c-ac')^2 = (bc'-b'c)(ab'-a'b)$$

1 common root

2 common roots $\rightarrow$ $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$