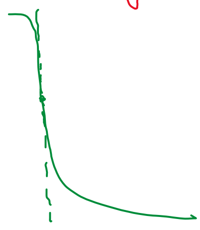
Differentiability

$$R[f'(x)] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad \lim_{h\to 0} \frac{f(x)-f(x-h)}{h}$$

Conditions for non-diff. (1) A corner (2) A cusp (3) Vertical tangent

(4) Discontinuity



$$f(x) = \begin{cases} |x-1|\left([x]-x\right) & , x \neq 1 \\ 0 & , x = 1 \end{cases}$$

Check differentiability at $x = 1$

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{|h|\left([1+h]-(1+h)\right)}{h}$$

$$= \lim_{h\to 0} \frac{h(1-1-h)}{h} = 0$$

$$\lim_{h\to 0} \frac{f(1)-f(1-h)}{h} = -\lim_{h\to 0} \frac{|1-h-1|\left([1-h]-(1-h)\right)}{h}$$

$$= -\lim_{h\to 0} \frac{\cancel{h}(0-1+h)}{\cancel{h}}$$

$$= -\lim_{h\to 0} (h-1) = 1$$

Properties

Assume $f(x)$ & $g(x)$ differentiable at $x = a$

(1) $cf(x)$ diff. at $x = a$,

(2) $f(x) \pm g(x) \xrightarrow{d} x = a$

(3) $f(x) \cdot g(x) \xrightarrow{d} x = a$

(4) $f(x) / g(x) \xrightarrow{d} x = a$, provided $g(a) \neq 0$

HW

16. Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then test whether

(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is differentiable at $x = 0$

Questions

5. If $f(x) = \begin{cases} (\cos x)^{1/\sin x} & \text{for } x \neq 0 \\ k & \text{for } x = 0 \end{cases}$

The value of k, so that f is continuous at $x = 0$ is

(A) 0 **(B)** 1 **(C)** 1/2 **(D)** None of these

$$\lim_{x\to 0} (\cos x)^{1/\sin x} = \lim_{x\to 0} e^{1/\sin x \cdot (\cos x - 1)}$$

$$= \lim_{x\to 0} e^{\frac{-2\sin^2 x/2}{2\sin x/2 \cdot \cos x/2}}$$

$$= \lim_{x\to 0} e^{-\tan x/2} = 1$$

13. Check the function $f(x) = \lim_{x\to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ for continuity and differentiability at $x = 0$.

LHL: $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \underline{\lim_{h\to 0} f(-h)}$

$$= \lim_{h\to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0-1}{0+1} = -1$$

RHL: $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \underline{\lim_{h\to 0} f(h)}$

$$= \lim_{h\to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h\to 0} \frac{1 - 1/e^{1/h}}{1 + 1/e^{1/h}} = 1$$

$\therefore$ RHL $\neq$ LHL $\Rightarrow$ discontinuous at $x = 0$

$\Rightarrow$ not differentiable at $x = 0$

3. $\lim_{x\to 1} \dfrac{x \sin\{x\}}{x - 1}$, where $\{x\}$ denotes the fractional part of x, is equal to

(A) -1 **(B)** 0 **(C)** 1 **(D)** Does not exist

$\lim_{x\to 1} \{x\} \to \lim_{x\to 1} (x - [x])$

(does not exist)

$$\lim_{x\to 1^+} \frac{x \sin\{x\}}{x-1} = \lim_{x\to 1^+} \frac{x \sin\{x\}}{\{x\}} \cdot \frac{\{x\}}{x-1}$$

$$= \lim_{x\to 1^+} x \cdot \lim_{x\to 1^+} \frac{\sin\{x\}}{\{x\}} \cdot \lim \frac{x-1}{x-1}$$

$1 \times 1 \times 1 = 1$

$$\lim_{x\to 1^-} \frac{x \sin\{x\}}{x-1}$$

$$\lim_{x\to 1^-} \frac{x}{x-1} \cdot \lim_{x\to 1^-} \sin\{x\}$$

$-\infty \cdot \sin(1)$