Q. Consider a r.s $X_1, X_2, \dots, X_n \overset{iid}{\sim} f_\theta(x)$, where $\boxed{f_\theta(x) = \frac{1+\theta}{(x+\theta)^2}, x \in [1, \infty)}$

Find the BCR to test $\{H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ $[\theta_1 > \theta_0]$

Applying NP Lemma:

$$\boxed{\sum_{i=1}^{n} \ln\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)} > \frac{1}{2}\left\{\ln k + n \ln\left(\frac{1+\theta_0}{1+\theta_1}\right)\right\}$$

↳ cannot put in the form of a fn of sample obs only.

BCR does not exist.

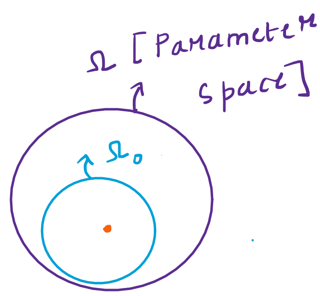
Likelihood Ratio Test

Extension of the NP Lemma for obtaining the BCR in the sense that composite hypothesis can also be incorporated.

Let $X_1, X_2, \dots, X_n \overset{iid}{\sim} f_\theta(x)$, θ = unknown popln parameter.

Let Ω = denote set of all possible of θ.

$\Omega_0 \subset \Omega$



To test: $H_0: \theta \in \Omega_0$ vs $\theta \notin \Omega_0$.

$\Rightarrow \theta \in (\Omega - \Omega_0)$

To find BCR for the test using Likelihood Ratio Test.

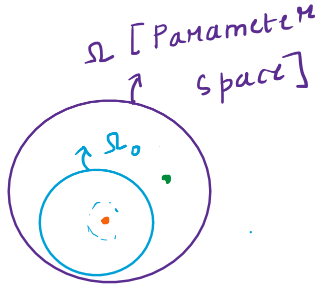
Define Likelihood fn: $L = \boxed{\prod_{i=1}^{n} f_\theta(x_i)}$

Under MLE: $\frac{\partial L}{\partial \theta} = 0 \Rightarrow$ solves for $\hat{\theta}_{MLE}$.

$$L = \boxed{\prod_{i=1}^{n} f_{\hat{\theta}_{MLE}}(x_i)} \Rightarrow \text{Maximized value of the Likelihood fn.}$$

$$L = \prod_{i=1} f_{\hat{\theta}_{MLE}}(x_i) \Rightarrow \text{Maximized value of the Likelihood fn.}$$

If $\hat{\theta}_{MLE} \in \Omega_0 \Rightarrow L(\hat{\theta}_{MLE})$ is the maximized value of likelihood fn $\forall\ \theta$.



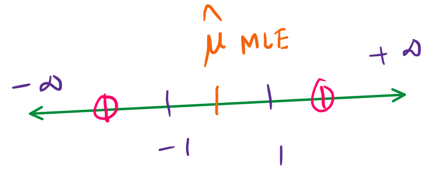
Define $\lambda = \dfrac{\text{Maximized } L \text{ for } \theta \in \Omega_0}{\text{Maximized } L \text{ for } \theta \in \Omega}$.

$\therefore$ Essentially $\Omega_0 \subset \Omega \Rightarrow 0 \leq \lambda \leq 1$.

Eg: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$

To test $H_0: \mu \in \underbrace{[-1, 1]}_{\Omega_0}$ vs $H_1: \mu \notin [-1, 1]$.

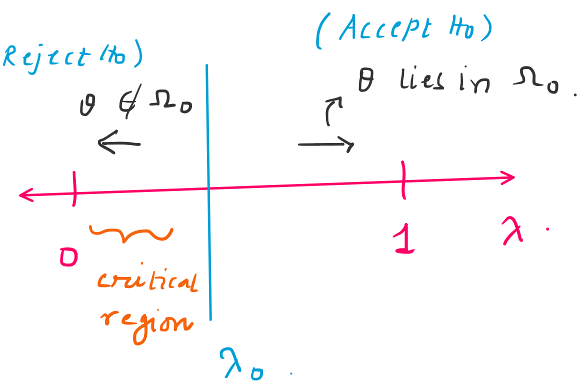
$\hat{\mu}_{MLE}$.



$$\lambda = \frac{\text{Max } L \text{ for } \mu \in [-1, 1]}{\text{Max } L \text{ for } \mu \in \mathbb{R}} \leq 1.$$

Compute a λ_0 s.t $\boxed{\lambda < \lambda_0} \Rightarrow$ BCR for the test. → Reject H_0.

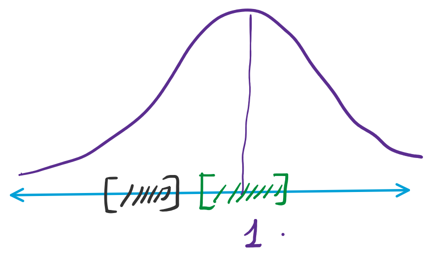
and λ_0 is computed based on L.O.S $= \alpha$ [fixed at the beginning of the test]



$$\boxed{P[\lambda < \lambda_0 \mid H_0] = \alpha} \Rightarrow \text{solve for } \lambda_0.$$

$H_0: \mu = 1$ vs $H_1: \mu = 2$

$\bar{x} > c \Rightarrow$ Reject H_0



Q. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where both μ, σ^2 are unknown.

Test: $H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$ vs $H_1: \mu \neq \mu_0,\ 0 < \sigma^2 < \infty$

Use the Likelihood ratio test to obtain the BCR.

HW Find $\hat{\mu}_{MLE}$, $\hat{\sigma}^2_{MLE}$.