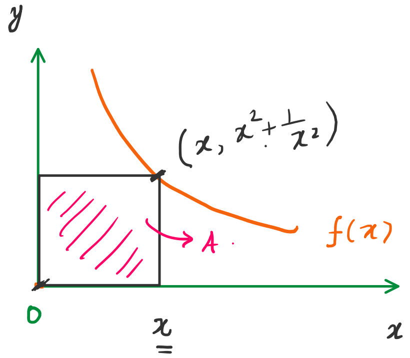
Q. A rectangle has its lower left hand corner at the origin and upper right hand corner on the graph $f(x) = x^2 + \frac{1}{x^2}$. Find the value of x so that area of rectangle is minimized.



$$f(x) = x^2 + \frac{1}{x^2}.$$

$$f'(x) = 2x - \frac{2}{x^3} = 2\left(x - \frac{1}{x^3}\right) \gtrless 0.$$

A = Area of the rectangle = $x f(x)$

$$A = x\left(x^2 + \frac{1}{x^2}\right)$$

$$A = x^3 + \frac{1}{x} \Rightarrow \text{Minimize.}$$

$\therefore$ For minimization: $\frac{dA}{dx} = 0 \Rightarrow 3x^2 - \frac{1}{x^2} = 0$

$$\Rightarrow 3x^2 = \frac{1}{x^2}$$

$$\Rightarrow x^4 = \frac{1}{3} \Rightarrow x = \left(\frac{1}{3}\right)^{1/4}.$$

Q. Let $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Determine the nature of the pt $x = 1$.

$$f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1.$$

$f'(x) = 4x^3 - 12x^2 + 12x - 4 \Rightarrow f'(1) = 0$ [Extremum].

$f''(x) = 12x^2 - 24x + 12 \Rightarrow f''(1) = 0$. [Extremum/ Pt of inflexion]

$f'''(x) = 24x - 24 \Rightarrow f'''(1) = 0$.

$f^{(IV)}(x) = 24 > 0$. [$x = 1$ is a pt of Minima].

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Q. Let $f: \mathbb{R} \to \mathbb{R}$ be an increasing fn s.t $f''(x) > 0$, $f(x) \neq 0$ and f^{-1} exists. Show that: $\frac{d^2\{f^{-1}(x)\}}{dx^2} < 0$.

Given $f(x)$, $f'(x) > 0$ and $f''(x) > 0$. , $f(x) \neq 0$.

Let $g(x) = f^{-1}(x)$. [Finding: $\frac{d^2\{f^{-1}(x)\}}{dx^2} = g''(x)$] .

$\Rightarrow \boxed{x = f\{g(x)\}}$ $\Rightarrow$ To find $g''(x)$.

Diff wrt x: $1 = f'\{g(x)\} \cdot g'(x)$ -

$$\Rightarrow g'(x) = \frac{1}{f'\{g(x)\}} > 0 \quad [\because f'(x) > 0 \ \forall x]$$

Diff wrt x: $$g''(x) = -\underbrace{\frac{1}{f'\{g(x)\}}}_{>0} \cdot \underbrace{f''\{g(x)\}}_{>0} \cdot \underbrace{g'(x)}_{>0} < 0.$$

$\hookrightarrow (\because f''(x) > 0 \ \forall x)$

$$\therefore g''(x) = \frac{d^2[f^{-1}(x)]}{dx^2} < 0 \text{ . [Proved] .}$$

Q. Let $f(x) = a_0 \cos|x| + a_1 \sin|x| + a_2|x|^3$. $f(x)$ is differentiable at $x = 0$ iff:

(a) $a_1 = 0$ & $a_2 = 0$

(b) $a_0 = 0$ & $a_1 = 0$

(c) $a_1 = 0$.

(d) $a_0, a_1, a_2 \in \mathbb{R}$.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} a_0 \cos x - a_1 \sin x - a_2 x^3, & x < 0 \\ a_0 \cos x + a_1 \sin x + a_2 x^3, & x \ge 0 \end{cases}$$

Check for diff at $x = 0$:-

$$LHD = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{a_0 \cos h - a_1 \sin h - a_2 h^3 - [a_0]}{h}$$

$$= \lim_{h \to 0^-} \frac{a_0 (\cos h - 1) - a_1 \sin h - a_2 h^3}{h}$$

$$= a_0 \cdot \underbrace{\lim_{h \to 0^-} \left(\frac{\cos h - 1}{h} \right)}_{= 0} - a_1 \cdot \underbrace{\lim_{h \to 0^-} \left(\frac{\sin h}{h} \right)}_{= 1} - a_2 \underbrace{\lim_{h \to 0^-} h^2}_{= 0}$$

$$= -a_1$$

HW

$$RHD = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{a_0 \cos h + a_1 \sin h + a_2 h^3 - a_0}{h}$$

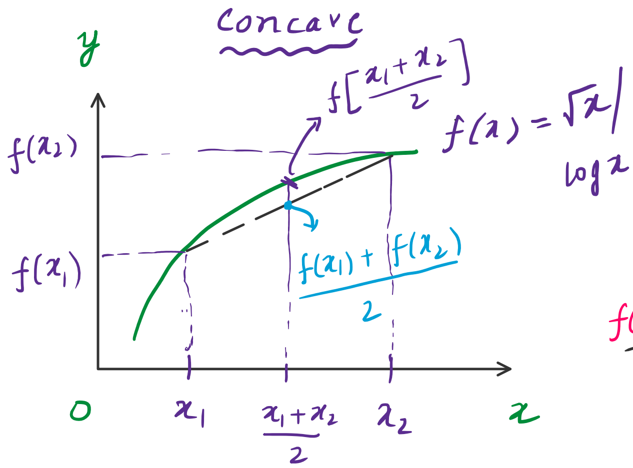
Q. Let $f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}$, where $x_1, x_2 \in \mathbb{R}$ and x_1, x_2 are independent. If $f(x)$ is differentiable and $f'(0) = a$, $f(0) = b$. Find : $d^k [f(x)]$

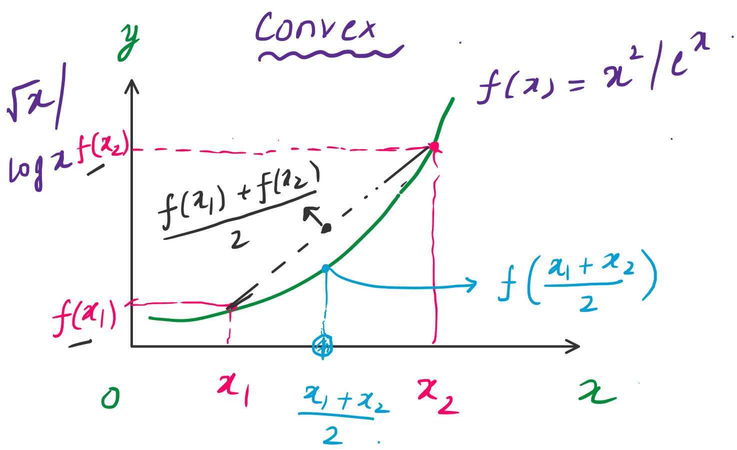
... x_1, x_2 are independent. If $f(x)$ is differentiable and $f'(0) = a$, $f(0) = b$. Find: $\dfrac{d^k[f(x)]}{dx^k}$ $(k \geq 4)$

Option 1: $f\left(\dfrac{x_1+x_2}{2}\right) = \dfrac{f(x_1)+f(x_2)}{2}$

Diff and get an idea about $f(x)$ ...

Option 2: Idea of concave and convex functions:-





$f'' < 0 \Rightarrow$ concave

$f\left[\dfrac{x_1+x_2}{2}\right] > \dfrac{f(x_1)+f(x_2)}{2}$

curve > Line.

$f'' > 0 \Rightarrow$ convex

$f\left[\dfrac{x_1+x_2}{2}\right] < \dfrac{f(x_1)+f(x_2)}{2}$.

Line > curve

Here,

$\therefore$ $f\left[\dfrac{x_1+x_2}{2}\right] = \dfrac{f(x_1)+f(x_2)}{2}$ ... St. line.

$\therefore$ $f(x) = mx + c$. $\Rightarrow$ $\dfrac{d^k[f(x)]}{dx^k} = 0$ $[k \geq 4]$.