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logarithms

Example 14. Find the least value of the expression $2\log_{10} x - \log_x 0.01$, where $x > 0, x \neq 1$.

$\log_x 0.01 = \log_x(10^{-2}) = -2\log_x 10$

$p = 2\log_{10} x + 2\log_x 10$

$\log_b a = \dfrac{1}{\log_a b}$

$= 2(\log_{10} x + \log_x 10)$

AM $\geq$ GM.

$\dfrac{\log_{10} x + \log_x 10}{2} \geq \sqrt{\log_{10} x \log_x 10}$

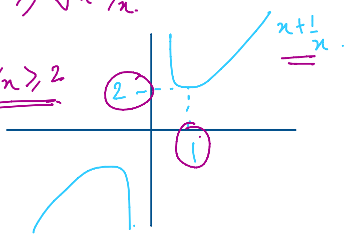
$\log_{10} x + \log_x 10 \geq 2.$

$p_{min} = 2 \times 2 = 4$

Which is smaller 2 or $(\log_\pi 2 + \log_2 \pi)$?

$\dfrac{x + 1/x}{2} \geq \sqrt{x \cdot 1/x}$

$x + 1/x \geq 2$



$y = x + \frac{1}{x}$

AM $\geq$ GM.

$\dfrac{x + 1/x}{2} \geq \sqrt{x \times \frac{1}{x}}$

≥ 1

$\boxed{x + 1/x \geq 2}$

Smaller = 2

x	$(x + 1/x)$
0.1	.
0.5	.
1	.
2	.
3	.

$p = \log_\pi 2 + \log_2 \pi$

AM $\geq$ GM.

$\dfrac{\log_\pi 2 + \log_2 \pi}{2} \geq \sqrt{\log_\pi 2 \log_2 \pi}$

$\log_\pi 2 + \log_2 \pi \geq 2.$

$\log_\pi 2 \neq 1$

$\log_\pi 2 = \dfrac{\log 2}{\log \pi} \neq 1$

$x + \frac{1}{x} \geq 2.$

when $x = 1$ $x + \frac{1}{x} = 2$

- **Ex. 1** *The expression* $\log_2 5 - \sum_{k=1}^{4} \log_2\left(\sin\left(\dfrac{k\pi}{5}\right)\right)$ $= 4$ *reduces to* $\dfrac{p}{q}$*, where p and q are co-prime, the value of* $p^2 + q^2$ *is*

$p = 4$ $q = 1$

$\sin(\pi - \theta) = \sin\theta$

to $\frac{p}{q}$, where p and q are co-prime, the value of $p^2 + q^2$ is

(a) 13 (b) 17 (c) 26 (d) 29

$\sin(\pi - \theta) = \sin\theta$

$$\log_2 5 - \left[\log_2 \sin\frac{\pi}{5} + \log_2 \sin\frac{2\pi}{5} + \log_2 \sin\frac{3\pi}{5} + \log_2 \sin\frac{4\pi}{5}\right]$$

$$= \log_2 5 - \left[\log_2\left(\sin\frac{\pi}{5} \cdot \sin\frac{2\pi}{5} \cdot \sin\frac{3\pi}{5} \cdot \sin\frac{4\pi}{5}\right)\right]$$

$\sin\frac{3\pi}{5} = \sin\left(\pi - \frac{2\pi}{5}\right) = \sin\frac{2\pi}{5}$

$\sin\frac{4\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{\pi}{5}$

$$= \log_2 5 - \log_2\left(\sin^2\frac{\pi}{5} \cdot \sin^2\frac{2\pi}{5}\right)$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$= \log_2 5 - \log_2\left[\left(\frac{1 - \cos\frac{2\pi}{5}}{2}\right)\left(\frac{1 - \cos\frac{4\pi}{5}}{2}\right)\right]$$

$\frac{\pi}{5} = 36^\circ$ $\quad \frac{2\pi}{5} = 72^\circ$

$$= \log_2 5 - \log_2\left[\frac{(1 - \cos 72^\circ)}{2}\frac{(1 - \cos 144^\circ)}{2}\right]$$

$\cos 72^\circ = \sin(90 - 72) = \sin 18^\circ$

$\cos 144 = \cos(180^\circ - 36^\circ) = -\cos 36^\circ$

$$= \log_2 5 - \log_2\left[\frac{(1 - \sin 18^\circ)}{2}\frac{(1 + \cos 36^\circ)}{2}\right]$$

$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$

$\cos 36^\circ = 1 - 2\sin^2 18^\circ = \frac{\sqrt{5}+1}{4}$

$$= \log_2 5 - \log_2\left[\frac{1 - \left(\frac{\sqrt{5}-1}{4}\right)}{2} \cdot \frac{1 + \left(\frac{\sqrt{5}+1}{4}\right)}{2}\right]$$

$$= \log_2 5 - \log_2\left[\left(\frac{5 - \sqrt{5}}{8}\right)\left(\frac{5 + \sqrt{5}}{8}\right)\right]$$

$$= \log_2 5 - \log_2\left(\frac{25 - 5}{8 \times 8}\right) = \log_2 5 - \log_2\left(\frac{\cancel{20}^{5}}{\cancel{64}_{16}}\right)$$

$$= \log_2\left(\frac{5}{5/16}\right) = \log_2 16 = 4$$

• **Ex. 3** *The lengths of the sides of a triangle are* $\log_{10} 12$, $\log_{10} 75$ *and* $\log_{10} n$, *where* $n \in N$. *If* a *and* b *are the least and greatest values of* n *respectively, the value of* $b - a$ *is divisible by*

(a) 221 (b) 222 (c) 223 (d) 224

difference of the other 2 sides < any side of a Δ < sum of the other 2 sides

$$\log_{10} 75 - \log_{10} 12 < \log_{10} n < \log_{10} 75 + \log_{10} 12$$

$$\log_{10}\left(\frac{75}{12}\right) < \log_{10} n < \log_{10}(75 \times 12)$$

$\frac{75}{12} < n < 75 \times 12$ $\qquad$ $75 \times 12 = 900$

$b - a = 892$ $\qquad$ $7 \leq n \leq 899$

a $\qquad$ b

• **Ex. 4** *If* $5\log_{abc}(a^3+b^3+c^3) = 3\lambda\left(\frac{1+\log_3(abc)}{\log_3(abc)}\right)$ *and* $\boxed{(abc)^{a+b+c} = 1}$ *and* $\lambda = \frac{m}{n}$, *where m and n are relative primes the value of* $|m+n| + |m-n|$ *is*

(a) 8 $\qquad$ (b) 10 $\qquad$ (c) 12 $\qquad$ (d) 14

$(abc)^{a+b+c} = 1 = abc^{0}$ $\qquad$ $abc \neq 1$

$a+b+c = 0$

$\boxed{a^3+b^3+c^3 = 3abc}$

LHS $= 5\log_{abc}(3abc)$

RHS $= 3\lambda\left[\frac{\log_3 3 + \log_3(abc)}{\log_3(abc)}\right]$

$= 3\lambda \frac{\log_3(3abc)}{\log_3(abc)} = 3\lambda \log_{abc}(3abc)$

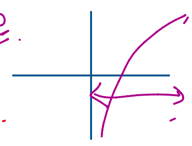
$m+n = 8$ $\qquad$ $3\lambda = 5$

$m-n = 2$ $\qquad$ $\lambda = \frac{5}{3}$ $\quad$ $5 \to m$, $3 \to n$

• *Ex. 6* *Number of real roots of equation* $3^{\log_3(x^2-4x+3)} = (x-3)$ *is*

(a) 0 $\qquad$ (b) 1 $\qquad$ (c) 2 $\qquad$ (d) infinite

$a^{\log_a b} = b$ $\Rightarrow b > 0$



(a) 0 (b) 1 (c) 2 (d) infinite

$$x^2 - 4x + 3 = x - 3$$

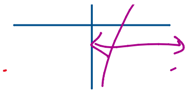
$$(x-3)(x-1) = (x-3)$$

$$(x-3)(x-1) - (x-3) = 0$$

$$(x-3)(x-2) = 0$$

$$x = 2, 3$$

$$a^{\log_a b} = b$$



$$x^2 - 4x + 3 > 0$$

$$(x-3)(x-1) > 0$$

$$x > 3 \text{ or } x < 1$$

$$\log x \Rightarrow x > 0$$

● **Ex. 8** *If* $x = \log_{2a}\left(\frac{bcd}{2}\right)$, $y = \log_{3b}\left(\frac{acd}{3}\right)$, $z = \log_{4c}\left(\frac{abd}{4}\right)$ *and* $w = \log_{5d}\left(\frac{abc}{5}\right)$ *and* $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd} N + 1$, *the value of* N *is*

(a) 40 (b) 80
(c) 120 (d) 160

$$x + 1 = 1 + \log_{2a}\left(\frac{bcd}{2}\right) = \log_{2a}\left(\frac{2abcd}{2}\right) = \log_{2a}(abcd)$$

$$\frac{1}{x+1} = \log_{abcd}(2a) \qquad \frac{1}{z+1} = \log_{abcd}(4c)$$

$$\frac{1}{y+1} = \log_{abcd}(3b) \qquad \frac{1}{w+1} = \log_{abcd}(5d)$$

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} + \frac{1}{w+1} = \log_{abcd}(2a \times 3b \times 4c \times 5d) = \log_{abcd}(120abcd)$$

$$= \log_{abcd}(120) + 1$$

Ex. 11 *Which of the following quantities are irrational for the quadratic equation*
$(\log_{10} 8)x^2 - (\log_{10} 5)x = 2(\log_2 10)^{-1} - x$?

(a) Sum of roots ✗ (b) Product of roots ✗
(c) Sum of coefficients ✓ (d) Discriminant ✓

$\left(\log 8\right)x^2 - \left(\log 5\right)x + x - 2\left(\log 2\right) = 0.$

$(\log 8)x^2 + (1-\log 5)x - \log 4 = 0.$

$(\log 8)x^2 + (\log 2)x - \log 4 = 0.$

$(3\log 2)x^2 + (\log 2)x - 2(\log 2) = 0$

$2\log 2$

sum of roots = $-1/3$
pdt of the roots = $-2/3$

$D = (\log 2)^2 + 24(\log 2)^2$
$= 25(\log 2)^2.$

Ex. 12 *The system of equations*
$\log_{10}(2000xy) - \log_{10} x \cdot \log_{10} y = 4$
$\log_{10}(2yz) - \log_{10} y \cdot \log_{10} z = 1$
and $\log_{10}(zx) - \log_{10} z \cdot \log_{10} x = 0$
has two solutions (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *then*

(a) $x_1 + x_2 = 101$ (b) $y_1 + y_2 = 25$
(c) $x_1 x_2 = 100$ (d) $z_1 z_2 = 100$

$\log_{10} x = a$
$\log_{10} y = b.$
$\log_{10} z = c$

$\log_{10} 2000 + a + b - ab = 4.$
$a + b - ab = 4 - 3 - \log 2$
$= 1 - \log 2 = \log 5.$

$\boxed{a + b - ab = \log 5}$ — ①

$\log 2 + b + c - bc = 1$

$b + c - bc = 1 - \log 2 = \log 5$

$\boxed{b + c - bc = \log 5}$ — ②

$\boxed{c + a - ac = 0.}$ — ③

$a + \cancel{b} - ab = \cancel{b} + c - bc.$

$(a - c) - ab + bc = 0$

$(a-c) - b(a-c) = 0.$

$(a-c)(1-b) = 0.$ $\Rightarrow$ $\boxed{a = c,\ b \neq 1}$

$a + 1 - a = \log 5$ ✗

$2a - a^2 = 0$

$a(2-a) = 0$

$\boxed{a = 0, 2}$

$\boxed{c = 0, 2}$

$\boxed{b = \log 5,\ \log 20}$

$\log_{10} x = 0, 2$ $\boxed{x = 1, 100}$

$\log_{10} z = 0, 2$ $\boxed{z = 1, 100.}$

$\log_{10} y = \log 5, \log 20$ $\boxed{y = 5, 20.}$

Suppose that $\log_{10}(x-2) + \log_{10} y = 0$ and
$$\sqrt{x} + \sqrt{(y-2)} = \sqrt{(x+y)}.$$

13. The value of x is
(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$ (c) $2\sqrt{2}$ (d) $4 - \sqrt{2}$

$\log_{10}[y(x-2)] = 0.$

$y(x-2) = 1$ — ①

$\cancel{x} + \cancel{y} - 2 + 2\sqrt{x(y-2)} = x + y$

$2\sqrt{x(y-2)} = 2.$

13. The value of x is

(a) $2+\sqrt{2}$ (b) $1+\sqrt{2}$ (c) $2\sqrt{2}$ (d) $4-\sqrt{2}$

14. The value of y is

(a) 2 (b) $2\sqrt{2}$ (c) $1+\sqrt{2}$ (d) $2+2\sqrt{2}$

15. If $x^{2t^2-6}+y^{6-2t^2}=6$, the value of $t_1\, t_2\, t_3\, t_4$ is

(a) 1 (b) 2 (c) 4 (d) 8

$\cancel{x}+\cancel{y}-2+2\sqrt{x(y-2)} = \cancel{x}+\cancel{y}$

$2\sqrt{x(y-2)} = 2.$

$x(y-2) = 1$ — (2)

$xy-2y=1$

$xy-2x=1$

$2y-2x=0$

$\boxed{x=y}$

$x = 1+\sqrt{2}.$

$x^2-2x-1=0$

$x = \dfrac{2\pm\sqrt{4+4}}{2} = \dfrac{2\pm2\sqrt{2}}{2} = 1\pm\sqrt{2}$

Ex. 25 *Prove that* $\log_3 5$ *is an irrational.*

- **Ex. 30** *Given,* $a^2 + b^2 = c^2$. *Prove that*

$$\log_{b+c} a + \log_{c-b} a = 2\log_{c+b} a \cdot \log_{c-b} a, \forall a > 0, a \neq 1$$

$$c - b > 0, c + b > 0$$

$$c - b \neq 1, c + b \neq 1.$$

• **Ex. 32** *If* $a^x = b, b^y = c, c^z = a, x = \log_b a^{k_1}, y = \log_c b^{k_2}$, $z = \log_a c^{k_3}$, *find the minimum value of* $3k_1 + 6k_2 + 12k_3$.

• **Ex. 33** *If* $x = 1 + \log_a bc, y = 1 + \log_b ca, z = 1 + \log_c ab$, prove that $xyz = xy + yz + zx$.

● **Ex. 36** *Find the square of the sum of the roots of the equation* $\log_2 x \cdot \log_3 x \cdot \log_5 x = \log_2 x \cdot \log_3 x + \log_3 x \cdot \log_5 x + \log_5 x \cdot \log_2 x$.

● **Ex. 37** *Given that* $\log_2 a = \lambda, \log_4 b = \lambda^2$ *and* $\log_{c^2}(8) = \dfrac{2}{\lambda^3 + 1}$, *write* $\log_2\left(\dfrac{a^2 b^5}{c^4}\right)$ *as a function of* 'λ' $(a, b, c > 0, c \neq 1)$.

- **Ex. 40** *Solve the following equations for x and y*

$\log_{100}|x+y|=\frac{1}{2}, \log_{10} y-\log_{10}|x|=\log_{100} 4.$