Q. Let $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where both μ, σ^2 are unknown.

Test: $H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$ vs $H_1: \mu \neq \mu_0,\ 0 < \sigma^2 < \infty$

Use the likelihood ratio test to obtain the BCR.

$X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, find the MLE of μ and σ^2.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}, \quad -\infty < x_i < \infty \ \forall i.$$

$\therefore$ Likelihood fn

$$L(\mu, \sigma^2 \mid \underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}.$$

Log-likelihood fn:

$$\ell(\mu, \sigma^2 \mid \underset{\sim}{x}) = -n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2.$$

$$= -n\ln\sigma - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2 + c.$$

For MLE:

$$\frac{\partial \ell}{\partial \mu} = 0 \Rightarrow 0 - \frac{1}{\cancel{2}\sigma^2}\cancel{2}\sum(x_i-\mu) = 0 \Rightarrow \sum(x_i-\mu) = 0 \Rightarrow \boxed{\hat{\mu}_{MLE} = \bar{x}}$$

$$\frac{\partial \ell}{\partial \sigma} = 0 \Rightarrow -\frac{n}{\sigma} - \frac{1}{\cancel{2}}\left(\frac{-\cancel{2}}{\sigma^3}\right)\sum(x_i-\mu)^2 = 0.$$

$$\Rightarrow \frac{n}{\sigma} = \frac{1}{\sigma^3}\sum(x_i-\mu)^2 \Rightarrow \boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i-\mu)^2}$$

↳ If μ is known.

But if μ is unknown $\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \hat{\mu})^2$

↳ If μ is known.

But, if μ is unknown, $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \hat{\mu}_{MLE})^2$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \bar{x})^2$$

To test: $H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$ vs $H_1: \mu \neq \mu_0,\ 0 < \sigma^2 < \infty$.

Ω_0.

$\Omega_0 = \{\mu = \mu_0,\ 0 < \sigma^2 < \infty\}$.

$\Omega = \{-\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$.

∴ Maximized value of L under $\boxed{\Omega}$

$= L(\hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE})$.

$\hat{\mu}_{MLE} = \bar{x}$

$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \bar{x})^2 = s^2$

$\sum(x_i - \bar{x})^2 = ns^2$

Here $L(\mu, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$

$$L(\hat{\mu}_{MLE}, \hat{\sigma}^2_{MLE}) = \frac{1}{(\hat{\sigma}\sqrt{2\pi})^n} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\hat{\mu})^2}$$

$$= \frac{1}{(\hat{\sigma}^2 2\pi)^{n/2}} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\hat{\mu})^2}$$

$$= \frac{1}{(\hat{\sigma}^2 2\pi)^{n/2}} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\bar{x})^2}.$$

$$\therefore L^{max}(\Omega) = \frac{1}{(s^2 2\pi)^{n/2}} e^{-n/2}$$

→ Maximized value of L under Ω.

To find the maximized value of L under $\Omega_0 = \{\mu = \mu_0, 0 < \sigma^2 < \infty\}$ (known)

$L^{max}(\Omega_0) = L(\mu_0, \hat{\sigma}^2_{MLE})$ [since μ is known under Ω_0]

$$= \frac{1}{\quad} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\mu_0)^2}$$

where, $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i - \mu_0)^2 = s^2$.

$$= \frac{1}{(\hat{\sigma}^2 2\pi)^{n/2}} e^{-\frac{1}{2\hat{\sigma}^2}\sum(x_i-\mu_0)^2}$$

where, $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i-\mu_0)^2 = s_0^2$.

$$= \frac{1}{(s_0^2 2\pi)^{n/2}} e^{-n/2}.$$

For Likelihood Ratio Test $\lambda = \frac{L^{max}(\Omega_0)}{L^{max}(\Omega)} = \left(\frac{s^2}{s_0^2}\right)^{n/2}$

For BCR under Likelihood Ratio Test, find $\boxed{\lambda < \lambda_0}$ => BCR for the test.

$$P[\lambda < \lambda_0 \mid H_0] = \alpha.$$

Q. Let $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Find the BCR using the likelihood ratio test for testing: $H_0: \mu = \mu_0,\ 0 < \sigma^2 < \infty$ vs $H_1: \mu > \mu_0,\ 0 < \sigma^2 < \infty$. where both μ, σ^2 are unknown.

$$\Omega = \{-\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}.$$

$$\Omega_0 = \{\mu = \mu_0,\ 0 < \sigma^2 < \infty\}$$

$$\hat{\mu}_{MLE} = \begin{cases} \bar{x}, & \mu > \mu_0 \\ \mu_0, & \mu \le \mu_0. \end{cases} \qquad \hat{\sigma}^2_{MLE} = \begin{cases} s^2, & \mu > \mu_0. \\ s_0^2, & \mu \le \mu_0 \end{cases}$$

where $s^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$, $s_0^2 = \frac{1}{n}\sum(x_i-\mu_0)^2$

HW. Find $L^{max}(\Omega)$, $L^{max}(\Omega_0)$, λ.