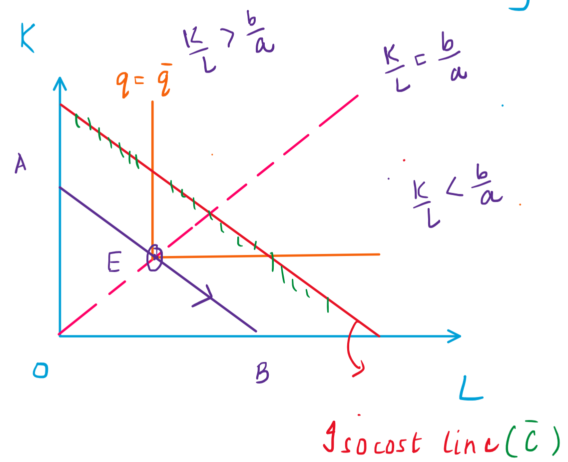
Q. Plot the isoquant and find the cost function for the following production functions:

(i) $q = \min\left\{\frac{L}{a}, \frac{K}{b}\right\}$, $a, b > 0$. [Leontief's production fn]

At opt: $\frac{L}{a} = \frac{K}{b} \Rightarrow \frac{K}{L} = \frac{b}{a}$.



Case I: $\frac{K}{L} > \frac{b}{a}$ or $\frac{K}{b} > \frac{L}{a}$.

$\therefore q = \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = \frac{L}{a}$.

$q = \frac{L}{a}$, Fix $q = \bar{q} \Rightarrow \bar{q} = \frac{L}{a}$

Diff: $d\bar{q} = \frac{1}{a} \cdot dL \Rightarrow dL = 0$.

Slope of isoquant $\frac{dK}{dL} \to \infty$. [vertical].

Case II: $\frac{K}{L} < \frac{b}{a} \Rightarrow \frac{K}{b} < \frac{L}{a}$.

$\frac{dK}{dL} = 0$. [horizontal].

Cost Function:

$\min_{\{L,K\}} wL + rK$ s.t $\bar{q} = q(L, K)$.

Cost exp $C = wL + rK$. [Combination of L, K that are affordable given $\bar{c}$].

Fix $C = \bar{c} \Rightarrow \bar{c} = wL + rK$ - - - Plot this eqn.

$\therefore d\bar{c} = w \cdot dL + r \cdot dK$.

$\frac{dK}{dL} = -\frac{w}{r} < 0$.

AB: Minimum possible isocost line.

$\therefore$ At opt: $\frac{L}{a} = \frac{K}{b} = q \Rightarrow L^* = aq$.

$\Rightarrow K^* = bq$

$\therefore$ Cost fn $= wL^* + rK^* = w \cdot aq + rbq$

$= (wa + rb)q$.

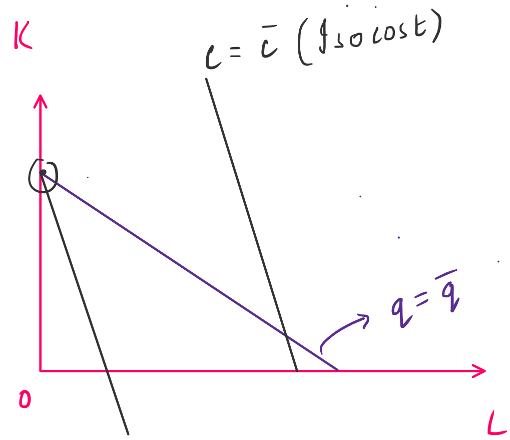
Q. $q = aL + bK, \quad a, b > 0$ .... [Perfect substitutes].

Fix $q = \bar{q} \Rightarrow \bar{q} = aL + bK$.

Diff $d\bar{q} = a \cdot dL + b \cdot dK$.

$d\bar{q} = 0 \Rightarrow a\,dL + b\,dK = 0$

$\Rightarrow \frac{dK}{dL} = -\frac{a}{b}$.



For cost fn: fix $q = \bar{q}$.

(i) Isocost steeper than isoquant.

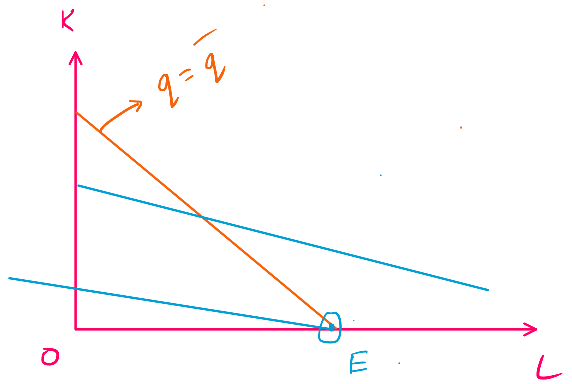
$L^* = 0$, $\bar{q} = bK \Rightarrow K^* = \frac{q}{b}$.

$C_1 = wL^* + rK^* = \frac{r \cdot q}{b}$ ✓

(ii) Isoquant steeper than isocost.

$K^* = 0$, $\bar{q} = aL \Rightarrow L^* = \frac{q}{a}$

$C_2 = wL^* + rK^* = \frac{wq}{a}$ ✓



Cost function: $C = \min\{C_1, C_2\}$

$= \min\left\{\frac{rq}{b}, \frac{wq}{a}\right\} = q \min\left\{\frac{w}{a}, \frac{r}{b}\right\}$

(iii) Isocost & Isoquant overlap

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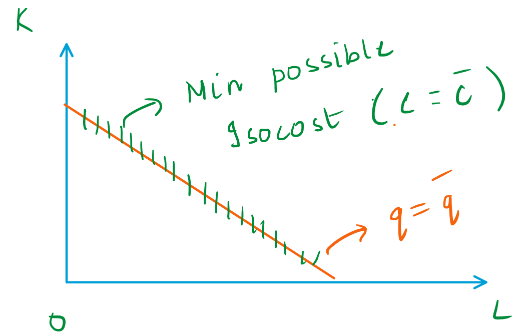
This need not be accounted for separately.



Q. $q = A L^{\alpha} K^{\beta}$ ---- [Cobb-Douglas production fn].

(i) A = level of technology.

α = share of labour in the production process $= \left(\frac{MP_L \cdot L}{q}\right)$

β = " " capital " " " " $= \left(\frac{MP_K \cdot K}{q}\right)$

$q = A L^{\alpha} K^{\beta}$, $\boxed{\frac{MP_L \cdot L}{q} = \alpha}$ => [derived].

(ii) Returns to Scale: Change in q, when L, K changed in same prop

$q = q(L, K) = A L^{\alpha} K^{\beta}$.

$$q(\lambda L, \lambda K) = A(\lambda L)^{\alpha}(\lambda K)^{\beta}$$
$$= \lambda^{\alpha+\beta} A L^{\alpha} K^{\beta} = \lambda^{\alpha+\beta} q(L, K)$$

$\therefore$ $q(\lambda L, \lambda K) = \lambda^{\alpha+\beta} q(L, K)$.

Case I: $\alpha + \beta > 1$ => Increasing Returns to scale (IRS)

Case II: $\alpha + \beta = 1$ => Constant Returns to scale (CRS)

Case III: $\alpha + \beta < 1$ => Decreasing Returns to scale (DRS).

Making isoquants:

Fix $q = \bar{q}$ => $\bar{q} = A L^{\alpha} K^{\beta}$.

Diff: $d\bar{q} = A\left[K^{\beta} \cdot \alpha \cdot L^{\alpha-1} dL + L^{\alpha} \cdot \beta K^{\beta-1} dK\right]$

$d\bar{q} = 0$ => $A\left[\alpha K^{\beta} L^{\alpha-1} dL + \beta L^{\alpha} K^{\beta-1} dK\right] = 0$.

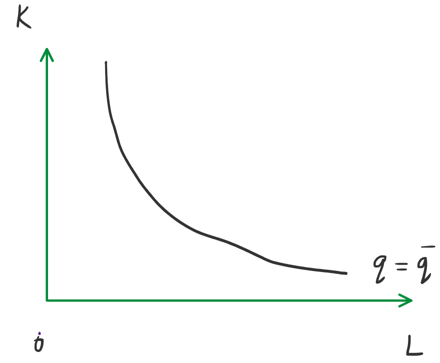
$A \neq 0$ => $\alpha \cdot K^{\beta} L^{\alpha-1} dL + \beta \cdot L^{\alpha} K^{\beta-1} dK = 0$.

$$\Rightarrow \left(\frac{\alpha}{L}\right) K^{\beta} L^{\alpha} \cdot dL + \left(\frac{\beta}{K}\right) \cdot L^{\alpha} K^{\beta} dK = 0.$$

$$\Rightarrow K^{\beta} L^{\alpha} \left[\left(\frac{\alpha}{L}\right) \cdot dL + \left(\frac{\beta}{K}\right) \cdot dK \right] = 0$$

$$K^{\beta} L^{\alpha} \neq 0 \quad \Rightarrow \quad \left(\frac{\alpha}{L}\right) dL + \left(\frac{\beta}{K}\right) \cdot dK = 0$$

$$\Rightarrow \quad \frac{dK}{dL} = -\frac{(\alpha/L)}{(\beta/K)} = -\left(\frac{\alpha}{\beta}\right)\left(\frac{K}{L}\right) < 0.$$



$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left[\frac{dK}{dL}\right] = \frac{d}{dL}\left[-\left(\frac{\alpha}{\beta}\right)\left(\frac{K}{L}\right)\right]$$

$$= -\left(\frac{\alpha}{\beta}\right) \cdot \frac{d}{dL}\left(\frac{K}{L}\right).$$

$$= -\left(\frac{\alpha}{\beta}\right)\left[\frac{L \cdot \frac{dK}{dL} - K}{L^2}\right]$$

$$= -\left(\frac{\alpha}{\beta}\right)\left[\frac{\cancel{L} \cdot \left(-\frac{\alpha}{\beta}\right)\left(\frac{K}{\cancel{L}}\right) - K}{L^2}\right]$$

$$= \left(\frac{\alpha}{\beta}\right)\left[\frac{\left(\frac{\alpha}{\beta}\right)K + K}{L^2}\right]$$

$$= \left(\frac{\alpha}{\beta}\right)\left[\frac{\alpha K + \beta K}{\beta L^2}\right] = \frac{\alpha K(\alpha+\beta)}{\beta^2 L^2} > 0.$$

[convex]