Random sample $X_1, X_2, X_3, \ldots, X_n \overset{iid}{\sim} f(\theta)$. [$\theta$ = unknown popln parameter].

Let T be the estimator used to get a robust estimate of θ.

Criteria for a Good Estimator:

(i) Unbiaseness: $E(T) = \theta$.

Unbiased estimators need not be unique.

$\therefore$ Out of all possible unbiased estimators, we choose the one with minimum possible variance.

The unbiased estimator is known as UMVUE. (Uniformly Minimum Var Unbiased Estimator)

Q. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Bernoulli(θ). Show that $\frac{T(T-1)}{n(n-1)}$ is an unbiased estimator of θ^2 where $T = \sum X_i$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} Ber(\theta)$.

$\therefore T = \sum X_i \sim Bin(n, \theta)$.

$E(T) = n\theta$.

$Var(T) = n\theta(1-\theta)$.

$E(T^2) = Var(T) + \{E(T)\}^2$

Check: $E\left[\frac{T(T-1)}{n(n-1)}\right] = \theta^2$.

LHS, $\frac{1}{n(n-1)} E[T^2 - T] = \frac{1}{n(n-1)}\{E(T^2) - \overset{n\theta}{E(T)}\}$.

$= \frac{1}{n(n-1)}\{n\theta(1-\theta) + n^2\theta^2 - n\theta\}$

$$= \frac{1}{n(n-1)} \left\{ n\theta(1-\theta) + n^2\theta^2 - n\theta \right\}$$

$$= \theta^2 \quad \text{[unbiased estimator]}.$$

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from Poisson (λ), $\lambda > 0$. Show that $\bar{X}$ is an UMVUE of λ.

Eg: $X \sim Poi(\lambda)$, $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots\ldots$

$$E(X) = \lambda = Var(X)$$

$$T = \bar{X} = \frac{1}{n}\sum X_i$$

$\therefore$ For unbiaseness: $E(\bar{X}) = E\left[\frac{1}{n}\sum X_i\right]$

$$= \frac{1}{n}\sum E(X_i) \quad [E(X_i) = \lambda \ \forall i]$$

$$= \frac{1}{n}\sum \lambda = \frac{n\lambda}{n} = \lambda.$$

$\left[Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{\lambda}{n} \right]$.

Note: For UMVUE, it is not possible to get all possible unbiased estimators and compare.

$\therefore$ Given the r.s., we compute Cramer-Rao Lower Bound (CRLB) [gives us the min possible variance for any unbiased estimator].

Note: $x_1, x_2, \ldots, x_n \overset{iid}{\sim} f(\theta)$

(i) Likelihood fn: $L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$ $\rightarrow$ $f_\theta(x)$.

(ii) Log-likelihood fn: $l(\theta) = \log L(\theta)$ $\rightarrow$ $l(\theta) = \log f_\theta(x)$

(ii) Log-likelihood fn: $\ell(\theta) = \log L(\theta)$ $\rightarrow$ $\ell(\theta) = \log f_\theta(x)$

(iii) Find: $g(\theta) = \frac{\partial}{\partial\theta} \cdot \ell(\theta)$. $\rightarrow$ $g(\theta) = \frac{\partial}{\partial\theta} \cdot \ell(\theta)$

(iv) Find $I(\theta) = E[g(\theta)]^2$. $\rightarrow$ $I(\theta) = E[g(\theta)]^2$

(v) $CRLB = \frac{1}{I(\theta)}$. $\rightarrow$ $CRLB = \frac{1}{I(\theta) \cdot n}$

Now, $f_\lambda(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$\ell(\lambda) = \log f_\lambda(x) = -\lambda + x \ln \lambda - \ln(x!) .$$

$$g(\lambda) = \frac{\partial}{\partial\lambda} \cdot \ell(\lambda) = -1 + \frac{x}{\lambda}.$$

$$I(\lambda) = E[g(\lambda)]^2 = E\left[\frac{x-\lambda}{\lambda}\right]^2 .$$

$$= \frac{1}{\lambda^2} E[x-\lambda]^2$$

$$= \frac{1}{\lambda^2} \cdot Var(x) = \frac{1}{\lambda^2} \cdot \lambda = \frac{1}{\lambda}.$$

$$\therefore CRLB = \frac{1}{I(\lambda) \cdot n} = \left(\frac{\lambda}{n}\right) = Var(\bar{X})$$

$\therefore$ $\bar{X}$ is the UMVUE for λ.

(ii) Consistency:

$\therefore$ An estimator T is said to be a consistent estimator of unknown popln parameter θ, if as $n \to \infty$

– × $E(T) \to \theta$ and $Var(T) \to 0$.

- $E(T) \to \theta$ and $Var(T) \to 0$.

[T converges to θ in probability] $P[|T-\theta| < \varepsilon] \to 1$ as $n \to \infty$.

Note: Let X_n, Y_n be consistent estimators s.t.

$X_n \xrightarrow{P} \theta_1$ and $Y_n \xrightarrow{P} \theta_2$.

Then, i). $\dfrac{X_n}{Y_n} \xrightarrow{P} \dfrac{\theta_1}{\theta_2}$ $[\theta_2 \neq 0]$

ii) $\alpha X_n + \beta Y_n \xrightarrow{P} \alpha \cdot \theta_1 + \beta \cdot \theta_2$.

iii) $X_n \cdot Y_n \xrightarrow{P} \theta_1 \cdot \theta_2$.

HW.

Q. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Let $s^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$

Show that s^2 is a consistent estimate of σ^2.