Construct estimators (to estimate under popln parameters)

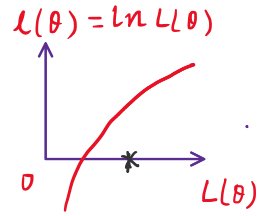
↳ Discuss their desirable properties [Unbiasedness, Minimum Variance, consistent estimator].

↳ Method of Estimation [Method of Moments, Method of Maximum Likelihood].

Maximum Likelihood Estimation:-

Let $X_1, X_2, \ldots, X_n$ be a r.s from $f_\theta(x)$. θ is the unknown population parameter.

We want to construct an estimator $\hat{\theta}_{MLE} = f(X_1, X_2, \ldots, X_n)$ using the method of maximum likelihood.



(i) Given the r.s, construct the likelihood fn:-

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \Rightarrow \text{Log-likelihood fn } l(\theta) = \ln L(\theta).$$

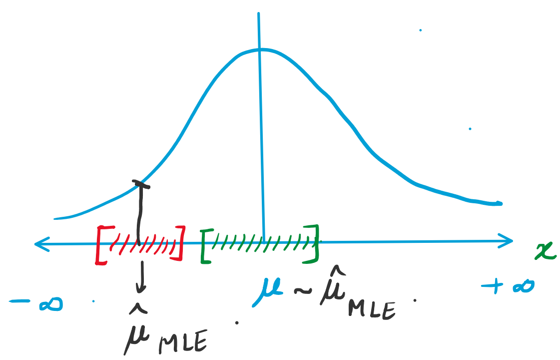
(ii) Find the value of 'θ', the maximizes the likelihood $[l(\theta)]$ of the sample:-

$$\frac{\partial \cdot L(\theta)}{\partial \theta} = 0 \Rightarrow \text{solving this gives } \hat{\theta}_{MLE}.$$

Note: Maximum likelihood estimators need not necessarily be unbiased.

Popln is $N(\mu, \sigma^2)$.

μ = unknown, σ^2 = known.



Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $P(\lambda)$. Find $\hat{\lambda}_{MLE}$.

pmf of Poisson distribution: $f_\lambda(x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

(i). $L(\lambda) = \prod_{i=1}^{n} f_\lambda(x_i) = \prod_{i=1}^{n} \dfrac{e^{-\lambda} \cdot \lambda^{x_i}}{(x_i)!} = \dfrac{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} (x_i)!}$

(ii) $l(\lambda) = \ln L(\lambda) = -n\lambda + \ln \lambda (\Sigma x_i) - \ln \left(\prod (x_i)!\right)$

(iii) $\dfrac{\partial l(\lambda)}{\partial \lambda} = 0 \Rightarrow -n + \dfrac{\Sigma x_i}{\lambda} = 0$.

$$\Rightarrow \frac{\Sigma x_i}{\lambda} = n \Rightarrow \boxed{\hat{\lambda}_{MLE} = \bar{x}} \rightarrow \text{unbiased estimate of } \lambda.$$

Q. Let $X_1, X_2, \ldots, X_n$ be a r.s from $N(\mu, \sigma^2)$ where both μ, σ^2 are unknown. Find $\hat{\mu}_{MLE}$, $\hat{\sigma}^2_{MLE}$.

pdf of $N(\mu, \sigma^2)$: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$; $x \in \mathbb{R}$.

(i) $L(\mu, \sigma^2) = \prod_{i=1}^{n} f_{\mu,\sigma^2}(x_i) = \prod_{i=1}^{n} \dfrac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$

$$= \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2}\right]\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2}\right] \cdots \left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}\right]$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\left[(x_1-\mu)^2 + (x_2-\mu)^2 + \cdots + (x_n-\mu)^2\right]}$$

$$= \frac{1}{} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}.$$

$$(\sigma\sqrt{2\pi})^n$$

(ii) Log-likelihood fn:

$$l(\mu,\sigma^2) = \ln L(\mu,\sigma^2)$$

$$= -n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$= -n\ln\sigma - \underbrace{n\ln(\sqrt{2\pi})}_{=k} - \dots$$

$$= -\frac{n}{2}\ln\sigma^2 + k - \dots$$

$$= k - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

(iii) $\frac{\partial l}{\partial\mu} = 0 \Rightarrow -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(x_i-\mu)(-1) = 0.$

$$\Rightarrow \frac{1}{\sigma^2}\sum(x_i-\mu) = 0$$

$$\Rightarrow \sum(x_i-\mu) = 0 \quad [\because \sigma^2 \neq 0]$$

$$\Rightarrow \sum x_i - n\mu = 0$$

$$\Rightarrow \boxed{\hat{\mu}_{MLE} = \bar{x}}$$

$\frac{\partial l}{\partial\sigma} = 0 \Rightarrow -\frac{n}{\cancel{2}}\cdot\frac{1}{\sigma^{\cancel{2}}}\cdot\cancel{2}\cancel{\sigma} + \frac{1}{\cancel{2}}\cdot\frac{(\cancel{2})}{\sigma^3}\cdot\sum(x_i-\mu)^2 = 0.$

$$\Rightarrow -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2 = 0.$$

$$\Rightarrow \frac{n}{\sigma} = \frac{1}{\sigma^3}\cdot\sum(x_i-\mu)^2$$

$\Rightarrow \sigma^2 = \frac{1}{n}\sum(x_i-\mu)^2$ [not MLE as μ is unknown].

$$\therefore \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(x_i-\hat{\mu}_{MLE})^2$$

$= \frac{1}{n}\sum(x_i-\bar{x})^2$ ...... [Biased estimate of σ^2]