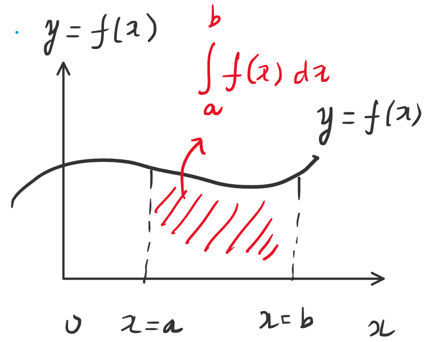
Integral Calculus

Eg: $\int f(x)\,dx$: Indefinite integrals [put 'c'].

$\int_a^b f(x)\,dx$: Definite integral [do not to put 'c'].

↳ Graphically: area under the curve from pt $x=a$ to pt $x=b$.



By defn: if $\int f(x)\,dx = \phi(x)+c$

then: $\int_a^b f(x)\,dx = \left[\phi(x)+c\right]_a^b$

$= \left[\phi(b)+c\right] - \left[\phi(a)+c\right]$

$= \phi(b) - \phi(a)$

Properties of Definite Integrals:

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \; ; \; a<c<b$ -- [Discontinuous fns]

(iii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(v) $\int_{-a}^{+a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x)=f(x) \Rightarrow \text{Even fn} \\ 0 & \text{if } f(-x) = -f(x) \Rightarrow \text{Odd fn.} \end{cases}$

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$$\int_{-\pi/2}^{\pi/2} \underbrace{[f(x)+f(-x)][g(x)-g(-x)]}\,dx .$$

Let $h(x)=[f(x)+f(-x)][g(x)-g(-x)]$

Check $h(-x) = [f(-x)+f(x)]\underbrace{[g(-x)-g(x)]}$

$$= -[f(x)+f(-x)][g(x)-g(-x)]$$

$$= -h(x)$$

$h(-x) = -h(x) \Rightarrow h(x)$ is odd:

$$\int_{-\pi/2}^{\pi/2} h(x)\,dx = 0 .$$

Q. Evaluate: $\displaystyle\int_0^x \frac{(t-|t|)^2}{1+t^2}\,dt,\ x\in\mathbb{R}.$

$$|t| = \begin{cases} t, & t\ge 0 \\ -t, & t<0 . \end{cases}$$

If $x \ge 0 \Rightarrow |t| = t > 0$.

$x<0 \Rightarrow |t| = -t$

Case I: If $x\ge 0$, $\Rightarrow |t| = t$

$$\int_0^x \frac{(t-t)^2}{1+t^2}\,dt = 0 .$$

Case II: If $x<0 \Rightarrow |t|=-t$

$$\int_0^x \frac{(t+t)^2}{1+t^2}\,dt = \int_0^x \frac{4t^2}{1+t^2}\,dt .$$

$$= 4\int_0^x \frac{t^2+1-1}{1+t^2}\,dt$$

$$= 4\int_0^x 1 - \frac{1}{1+t^2}\, dt$$

$$= 4\left[t - \tan^{-1} t\right]_0^x$$

$$= 4\left\{\left[x - \tan^{-1}x\right] - \left[0-0\right]\right\}$$

$$= 4\left[x - \tan^{-1}x\right]$$

$$I = \begin{cases} 0, & x \geq 0 \\ 4\left[x - \tan^{-1}x\right], & x < 0 \end{cases}$$

Q. If $n \neq 1$, Find: $\displaystyle\int_0^{\pi/4} \left(\tan^n x + \tan^{n-2} x\right) d\left(x - [x]\right)$

$$d\left(x - [x]\right) = dx - d[x] = dx - d(0) = dx$$

$x \in \left[0, \frac{\pi}{4}\right]$

$$\frac{\pi}{4} = \frac{22/7}{4} = \frac{22}{28} < 1$$

$$\int_0^{\pi/4} \left(\tan^n x + \tan^{n-2} x\right) dx$$

$$\int_0^{\pi/4} \tan^{n-2} x \left(1 + \tan^2 x\right) dx$$

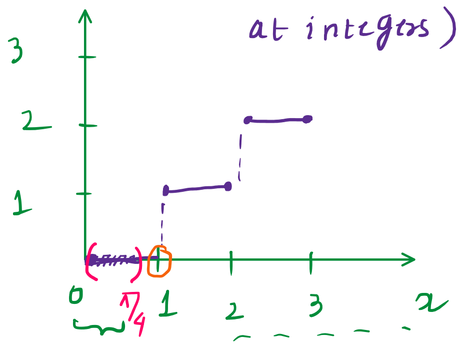
$$\int_0^{\pi/4} \tan^{n-2} x \cdot \sec^2 x \, dx$$

Let $\tan x = z \Rightarrow \sec^2 x\, dx = dz$

$x = 0,\ z = 0$

$x = \pi/4,\ z = 1$

$$\int_0^1 z^{n-2}\, dz = \left[\frac{z^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$$



(Discontinuous at integers)

$[x] = 0, \quad 0 \leq x < 1$

$\quad = 1, \quad 1 \leq x < 2$

$\quad = 2, \quad 2 \leq x < 3$

$x \in \left[0, \frac{\pi}{4}\right],\ [x] = 0$

Find $\displaystyle\int_0^{1.3} [x]\, dx$

$$= \int_0^1 [x]\, dx + \int_1^{1.3} [x]\, dx$$

$$= \int_0^1 0\, dx + \int_1^{1.3} 1\, dx$$

$[1.2] = 1$

HW

Q. Let $f(x)$ be s.t $f(x) = f'(x)$ with $f(0) = 1$ and $g(x)$ be another fn s.t $f(x) + g(x) = x^2$. Find:

$$\int_0^1 f(x) \cdot g(x)\, dx.$$