# Second-order Differential Equations.

## Linear Differential Equation of Second Order:

Recap: Linear Differential Equation [First order]

Form: $\frac{dy}{dx} + P \cdot y = Q$ [where P, Q are fns of $x$]

For Linear Differential Equation [Second Order].

(*) Form: $\frac{d^2y}{dx^2} + P_1 \cdot \frac{dy}{dx} + P_2\, y = Q$ [where $P_1, P_2, Q$ are fns of $x$]

## Linear Differential Equation with constant coeffecients:

Form: $\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2\, y = Q$ [ $P_1, P_2$ are constant
Q is a fn of Q ]

If $Q = 0 \Rightarrow$ RHS of diff eqn $= 0 \Rightarrow$ Homogeneous diff eqn.

If $Q \neq 0 \Rightarrow$ Non-homogeneous.

Note: i) First solve the homogeneous part. [Complementary fn (CF)]

ii) Then include Q to find the Particular Integral (PI)

Final soln: $y = CF + PI$

## Solution technique:

Homogeneous diff eqn: $\frac{d^2y}{dx^2} + P_1 \cdot \frac{dy}{dx} + P_2\, y = 0$ --- (i)

→ unknown is 'm'.

Begin with a trial soln: Let $y = e^{mx}$ be a trial soln.

∴ Trial soln must satisfy eqn (i).

$\frac{dy}{dx} = m e^{mx}$ $\frac{d^2y}{dx^2} = m^2 e^{mx}$

Put in (i):

$m^2 e^{mx} + P_1 (m e^{mx}) + P_2 e^{mx} = 0$.

Homo 1st order Diff Eqn:

$\frac{dy}{dx} + Py = 0$.

$\frac{dy}{dx} = -Py$.

$\int \frac{dy}{y} = -\int P \cdot dx$

$$m^2 e^{mx} + P_1 (m e^{mx}) + P_2 e^{mx} = 0 .$$

$$e^{mx} [m^2 + P_1 \cdot m + P_2] = 0$$

$$e^{mx} \neq 0 \Rightarrow \boxed{m^2 + P_1 m + P_2 = 0} \text{ -- Auxiliary Eqn (AE)}$$

↳ 2 solns of 'm'

$$\int \frac{dy}{y} = -\int P \cdot dx$$

$$\ln y = -\int P dx .$$

$$\ln y = \phi(x) + c .$$

$$y = e^{\phi(x) + c} .$$

$$y = e^{c} \cdot e^{\phi(x)}$$

$$y = k e^{\phi(x)}$$

∴ 3 Possible Nature of Roots :-

(i) Real & Unequal $[m_1, m_2 ; m_1 \neq m_2]$

$\rightarrow y = e^{m_1 x} \rightarrow$ First fn

$\rightarrow y = e^{m_2 x} \rightarrow$ second soln

Then $y = \boxed{c_1 f_1 + c_2 f_2}$

C.F: $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(ii) Real & Equal $[m_1 = m_2 = m \text{ (say)}]$

C.F: $y = c_1 e^{mx} + c_2 x e^{mx} = (c_1 + c_2 x) e^{mx} .$

(iii) Complex Roots $[m_1, m_2 = \alpha \pm i\beta]$

C.F: $y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

Q. Solve: $\dfrac{d^2y}{dx^2} - 5 \dfrac{dy}{dx} + 6y = 0$ ---- (i)

Let $y = e^{mx}$ be a trial soln

Put the trial soln in (i):

AE: $m^2 - 5m + 6 = 0$

$m = 2, 3 .$

$y = c_1 e^{2x} + c_2 e^{3x}$ [$c_1, c_2$ are constants] .

Q. Solve: $\dfrac{d^2y}{dx^2} + y = 0 .$

AE: $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm\sqrt{-1} = \pm i$

compare: $\alpha = 0, \beta = 1 .$

$y = e^{0x} [c_1 \cos x + c_2 \sin x]$

$= c_1 \cos x + c_2 \sin x$ [$c_1, c_2$ are arbitrary constant]

Q. Let $y(x)$ be a non-trivial soln of the eqn:

Q. Let $y(x)$ be a non-trivial soln of the eqn:

$$\frac{d^2y}{dx^2} + 2c\cdot\frac{dy}{dx} + ky = 0;\ c<0,\ k>0,\ c^2>k. \text{ Then:}$$

(a) $|y(x)| \to \infty$ as $x \to \infty$.

(b) $|y(x)| \to 0$ as $x \to \infty$

(c) $\lim_{x\to\infty} |y(x)|$ exists & its finite

(d) None.

$$\frac{d^2y}{dx^2} + 2c\cdot\frac{dy}{dx} + ky = 0$$

Trial soln: $y = e^{mx}$.

AE: $m^2 + 2cm + k = 0 \Rightarrow m = \dfrac{-2c \pm \sqrt{4c^2 - 4k}}{2}$

$$m = \frac{-\cancel{2}c \pm \cancel{2}\sqrt{c^2-k}}{\cancel{2}} = -c \pm \underbrace{\sqrt{c^2-k}}_{>0} = a \Rightarrow \text{Real \& unequal Roots.}$$

$m = -c \pm a$.

Soln: $y = c_1 e^{(-c+a)x} + c_2 e^{(-c-a)x}$.

$y = e^{-cx}\left[c_1 e^{ax} + c_2 e^{-ax}\right]$. $e^{-cx} \to \infty$, as $x \to \infty$,

$\Rightarrow y \to \infty$ (a)

Note: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

If $m_1 < 0,\ m_2 < 0$

$\lim_{x\to\infty} c_1 e^{m_1 x} \to 0$, $\lim_{x\to\infty} c_2 e^{m_2 x} \to 0$, $y \to 0$ as $x \to \infty$

If $(m_1 < 0,\ m_2 > 0)$ or $(m_1 > 0,\ m_2 < 0)$

$\Downarrow$

$\lim_{x\to\infty} c_1 e^{m_1 x} \to 0$, $\lim_{x\to\infty} c_2 e^{m_2 x} \to \infty$, $y \to \infty$.