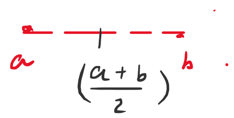
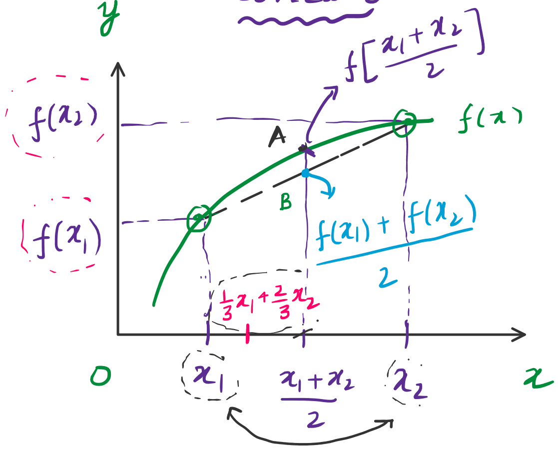
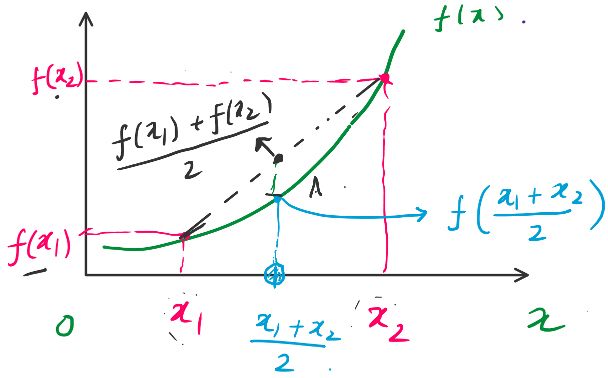
Concave and convex Functions



Concave



Convex



Concave fn: $f'' < 0$

Concave: $\frac{f(x_1)+f(x_2)}{2} < f\left[\frac{x_1+x_2}{2}\right]$

↳ line ↳ curve

Curve > Line

Convex fn: $f'' > 0$.

Convex: $\frac{f(x_1)+f(x_2)}{2} > f\left[\frac{x_1+x_2}{2}\right]$

Line > curve

Generalizing the idea of concavity and convexity:

(i) Consider $x_1, x_2 \in \mathbb{R}$. and fn values as: $f(x_1)$ and $f(x_2)$.

Take any convex combination $\theta x_1 + (1-\theta) x_2$, $0 \leq \theta \leq 1$.

Fn value: $f[\theta x_1 + (1-\theta) x_2]$, $0 \leq \theta \leq 1$ ---- (i)

Value on line: $\theta \cdot f(x_1) + (1-\theta) \cdot f(x_2)$, $0 \leq \theta \leq 1$ ---- (ii)

(*)

$\therefore$ Concavity: Curve > Line.

$$f[\theta x_1 + (1-\theta) x_2] > \theta \cdot f(x_1) + (1-\theta) \cdot f(x_2)$$

(*) Convexity: Line > curve.

$$\theta \cdot f(x_1) + (1-\theta) f(x_2) > f[\theta \cdot x_1 + (1-\theta) x_2].$$

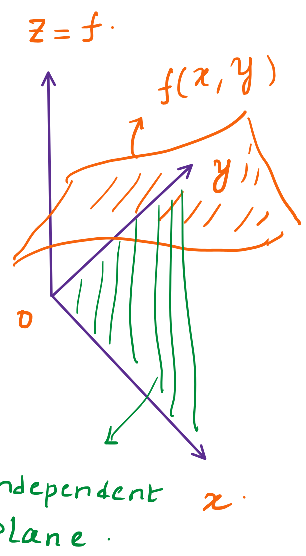
(ii) Concavity, convexity for f...

$z = f$.

(ii) Concavity, convexity for fn of 2-variables:

Suppose $f(x,y)$.

Here there are 2-independent variables x, y.



Consider (x_1, y_1) $(x_2, y_2) \in \mathbb{R}^2$ -

and fn value to be $f(x_1, y_1)$ and $f(x_2, y_2)$.

Convex combination of (x_1, y_1) and (x_2, y_2): $\theta\cdot(x_1, y_1) + (1-\theta)\cdot(x_2, y_2)$. , $0 \leq \theta \leq 1$.

Fn value : $f[\theta(x_1, y_1) + (1-\theta)(x_2, y_2)]$ .... (i)

Line : $\theta\cdot f(x_1, y_1) + (1-\theta)\cdot f(x_2, y_2)$ ---- (ii)

Concave: curve > Line. (i) > (ii) | Convex: Line > Curve (ii) > (i)

Taylor series Expansion:-

Consider a fn $f(x)$, whose properties are known at pt $x = a$.

$$\therefore \quad f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + R_n.$$

Expanding around pt $x = 0$, $[a = 0]$.

$$(*) \quad f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \qquad [*].$$

We obtain the infinite series expansion of fns:-

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Eg: $f(x) = e^x$.

$$e^x = e^0 + x e^0 + \frac{x^2}{2!} e^0 + \cdots\cdots = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\cdots$$

$$e^{-x} = \underbrace{e^{-0}}_{=1} + x(-\underbrace{e^{-0}}_{=1}) + \frac{x^2}{2!}(\underbrace{e^{-0}}_{=1}) + \cdots\cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots$$

$f(x) = e^{-x}$
$f'(x) = -e^{-x}$
$f''(x) = e^{-x}$
$f'''(x) = -e^{-x}$.

Note: We know the infinite series exp of $f(x)$. [given by (*)]
Then, to obtain expansion of $f(-x)$, replace all terms in the expansion of $f(x)$ by $(-x)$.

We know: $f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$ [**]

$$f(-x) = e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots$$

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \cdots\cdots$$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots ; -1 < x \leq 1$ [**]

$$\log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} - \cdots\cdots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots ; \quad -1 \leq x < 1$$

$$\log(x+2) = \log(2+x)$$
$$= \log\left\{2\left(1+\frac{x}{2}\right)\right\}.$$
$$= \log 2 + \underbrace{\log\left(1+\frac{x}{2}\right)}$$
$$= \log 2 + \left[\left(\frac{x}{2}\right) - \frac{(x/2)^2}{2} + \frac{(x/2)^3}{3} \cdots\right]; -1 < \frac{x}{2} \leq 1$$

$$\left.\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots \end{aligned}\right\} [**].$$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

[Replaced 'x' by $2x$ in expansion of $\sin x$]

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$= \frac{1}{2} - \frac{1}{2}\left[1 - \frac{(2x)}{2!} + \frac{(2x)^4}{4!} - \cdots\cdots\right].$$

Note: If we know the infinite series expansion of $f(x)$ given by [*], then we can obtain the expansion of $f(kx)$ / $f(x+k)$, k is a constant [by replacing x by kx / $(x+k)$ in the exp of $f(x)$].

But we cannot use the exp of $f(x)$ to obtain expansion of $\{f(x)\}^k$ directly. We need to apply some transformations to use exp of $f(x)$.