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Chebyshev's Inequality

role of std deviation as a parameter to characterise variance interpreted by means.

Theorem: If x is a random variable with mean μ and variance σ^2, then for any positive number k, we have

$$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$E(x) = \Sigma x \cdot f(x)$
$E(x) = \int x \cdot f(x)\,dx$

or,

$$P\{|x-\mu| < k\sigma\} \leq 1 - \frac{1}{k^2}$$

Proof: x is a continuous r.v. By def,

$$\sigma^2 = \sigma_x^2 = E(X-E(x))^2$$
$$= E(X-\mu)^2$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$, where $f(x)$ is a pdf of x.

$[-\infty$ to $\mu - k\sigma]$
$[(\mu - k\sigma)$ to $(\mu + k\sigma)]$
$[(\mu + k\sigma)$ to $\infty]$

$$\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx$$

$[(\mu-k\sigma)$ to $(\mu+k\sigma)]$

$[(\mu+k\sigma)$ to $\infty]$

$$\sigma^2 = \int_{-\infty} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$x \le \mu - k\sigma$ and $x \geq \mu + k\sigma$

Now we have

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{--- (1)}$$

$$x \le \mu - k\sigma \text{ and } x \geq \mu + k\sigma$$

$$\therefore \boxed{|x-\mu| \geq k\sigma} \quad \text{--- (2)}$$

$$\sigma^2 \geq k^2\sigma^2 \left[\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx\right]$$

$$\sigma^2 \geq k^2\sigma^2 \left[P(x \le \mu - k\sigma) + P(x \geq \mu + k\sigma)\right]$$

$$\sigma^2 \geq k^2\sigma^2 P(|x-\mu| \geq k\sigma)$$

$$\Rightarrow \boxed{P(|x-\mu| \geq k\sigma) \le \frac{1}{k^2}}$$

or, $P\{|x-\mu| < k\sigma\} = 1 - P\{|x-\mu| \geq k\sigma\} \geq 1 - \dfrac{1}{k^2}$

Generalised form of Chebychev's Inequality:

Let $g(x)$ be a non-negative function of a rv X. Then for every $k>0$, we have,

$$P\{g(x) \geq k\} \leq \frac{E(g(x))}{k}$$

Let S be the set of all x where $g(x) \geq k$ ie,

$$S = \{x : g(x) \geq k\}$$

we define c.d.f

then $\int_S dF(x) = P(X \in S) = P[g(x) \geq k]$

where $F(x)$ is the distribution fn of X.

Now, $E(g(x)) = \int_{-\infty}^{\infty} g(x)\, dF(x) \geq \int_S g(x)\, dF(x)$

$\geq k \cdot P[g(x) \geq k]$

$$\Rightarrow \boxed{P[g(x) \geq k] \leq \frac{E(g(x))}{k}}$$

(Proved).

If we take $g(x) = \{X - E(x)\}^2 = (X-\mu)^2$

If we take $g(x) = \{x - E(x)\}^2 = (x-\mu)^2$
and replace k by $k^2\sigma^2$, we get

$$P\{(x-\mu)^2 \geq k^2\sigma^2\} \leq \frac{E(\cancel{x}-\mu)^2}{k^2\sigma^2} = \frac{\cancel{\sigma^2}}{k^2\cancel{\sigma^2}} = \frac{1}{k^2}$$

$$\Rightarrow \boxed{P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}}$$ ✓

which is Chebychev's inequality.

Let us take $g(x) = |x|$, for any $k > 0$

$$P[|x| \geq k] \leq \frac{E|x|}{k}$$

which is Markov's Inequality

Now let us take $g(x) = |x|^r$ and replacing k by k^r we get a more generalised inequality

$$P[|x|^r \geq k^r] \leq \frac{E|x|^r}{k^r}$$

This is generalised Markov's inequality

Convergence in Probability:

A sequence of random variables $X_1, X_2, X_3, \dots X_n$ is said to converge in probability to a constant a if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - a| < \varepsilon) = 1$$

or $$\lim_{n \to \infty} P(|X_n - a| \geq \varepsilon) = 0$$

$$\therefore \boxed{X_n \xrightarrow{P} a \text{ as } n \to \infty}$$

If there exist a rv X such that $X_n - X \xrightarrow{P} a$ as $n \to \infty$ then we say that the given sequence of rv converges in probability to the rv X.

Simple rules for convergence in probability,

If $X_n \xrightarrow{P} \alpha$ and $Y_n \xrightarrow{P} \beta$ as $n \to \infty$ then

(i) $X_n \pm Y_n \xrightarrow{P} \alpha \pm \beta$ as $n \to \infty$

(ii) $X_n Y_n \xrightarrow{P} \alpha . \beta$ as $n \to \infty$

(iii) $\dfrac{X_n}{Y_n} \xrightarrow{P} \dfrac{\alpha}{\beta}$ as $n \to \infty$

provided $\beta \neq 0$

CHEBYCHEV'S THEOREM:

Immediate consequence of Chebychev's inequality we have the following theorem and convergence in probability

"If $x_1, x_2 \ldots \ldots x_n$ is a sequence of r.v and if mean μ_n and standard deviation σ_n of X_n exists for all n and if $\sigma_n \to 0$ as $n \to \infty$ then

$$X_n - \mu_n \xrightarrow{P} 0 \quad \text{as } n \to \infty."$$

Proof: We know, for any $\varepsilon > 0$

$$P\{|X_n - \mu_n| \geq \varepsilon\} \leq \frac{\sigma_n^2}{\varepsilon^2} \to 0 \text{ as } n \to \infty$$

Hence $X_n - \mu_n \xrightarrow{P} 0$ as $n \to \infty$

(proved)

Weak Law of Large Numbers (W.L.L.N)

Let $X_1, X_2, X_3, \ldots\ldots, X_n$ be a sequence of random variables and $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ be their respective expectations and let $B_n = Var(X_1 + X_2 + \cdots + X_n) < \infty$

and let $B_n = Var(x_1 + x_2 + \cdots + x_n) -$

Then $$P\left\{\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - \frac{\mu_1 + \mu_2 + \cdots + \mu_n}{n}\right| < \varepsilon\right\} \geq 1 - \eta$$

for all $n > n_0$, where ε and η are arbitrary small positive numbers, provided $\lim_{n \to \infty} \frac{B_n}{n^2} \to 0$.

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