Number Systems

Tuesday, March 14, 2023 8:00 PM

$\{(2n+1),\ n \in \mathbb{N}\} \rightarrow$ odd

even + 1 = odd

$\underbrace{(2n+1)^2}_{\text{odd}^2} = 4n^2+4n+1 = \underbrace{4(n^2+n)}_{\text{even}}+1$

EXAMPLE 1. *If (a/b) < (c/d) with b > 0, d > 0 show that (a + c)/(b + d) lies between a/b and c/d. (where a, b, c, d are real numbers).*

$\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc \Rightarrow ab + ad < ab + bc$

$\Rightarrow a(b+d) < b(a+c)$

$\Rightarrow \frac{a}{b} < \frac{a+c}{b+d}$ —— (1)

To show

$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$

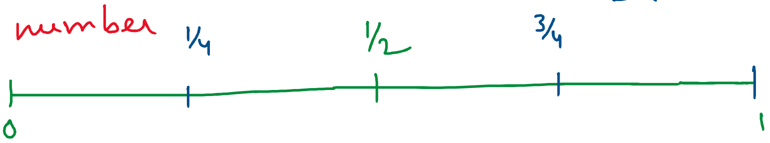
$ad < bc \Rightarrow cd + ad < cd + bc$

$\Rightarrow \frac{a+c}{b+d} < \frac{c}{d}$ —— (2)

Implies → between 2 real numbers, there is another real number

$a=2, b=1, c=3, d=1$

$\frac{2+3}{1+1} = \frac{5}{2} = 2\frac{1}{2}$



$R = \{\frac{1}{n},\ n \in \mathbb{N}\} \rightarrow (0,1]$ (range)

(unit fractions)

$Q + Q^c$

$R = \begin{cases} 1 \rightarrow 1 \\ 2 \rightarrow 1/2 \\ 3 \rightarrow 1/3 \end{cases}$ ↓ map

EXAMPLE 2. *Let a and b be positive integers. Show that √2 always lies between (a/b) and (a + 2b)/(a + b).*

✓ Let $\sqrt{2} < \frac{a}{b} \Rightarrow 2 < \frac{a^2}{b^2} \Rightarrow 2b^2 < a^2$ —— (1)

$a^2+4b^2 = a^2+2b^2+2b^2 < a^2+a^2+2b^2 = 2a^2+2b^2$

$\Rightarrow a^2+4b^2 < 2(a^2+b^2)$ —— (2)

$(a+2b)^2 = a^2+4ab+4b^2 < 2(a^2+b^2)+4ab$

$= 2(a^2+b^2+2ab)$

$\Rightarrow (a+2b)^2 < 2(a+b)^2$

$\Rightarrow \frac{a+2b}{a+b} < \sqrt{2}$ (Case 1)

To show:

(1) $\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$

(2) $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$

Reversing the steps, we get the second half.

$\sqrt{2} > \frac{a}{b} \Rightarrow a^2 < 2b^2$

✓ $2(a+b)^2 = 2(a^2+2ab+b^2) = a^2+a^2+4ab+4b^2 < 2b^2+a^2+4ab+4b^2$

sum of square numbers $\left\{ \begin{array}{l} = 2b^2 + (a+2b)^2 \\ < (a+2b)^2 \end{array} \right.$

$\Rightarrow 2(a+b)^2 < (a+2b)^2$

$\Rightarrow \sqrt{2} < \left(\frac{a+2b}{a+b}\right)$

Number Systems

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ex^i

EXAMPLE 3. *Given any real number $x > 0$, show that there exists an irrational number ξ, such that $0 < \xi < x$.*

Case 1 x is irrational, $\exists\, e,\ 0<e<x,\ e \in \mathbb{Q}^c$

Consider $x/2$. Set $e = x/2$, $\exists\, e, e \in \mathbb{Q}^c$, $0<e<x$

Case 2 x is rational, $\exists\, e,\ 0<e<x,\ e \in \mathbb{Q}^c$

Consider $x/\sqrt{2}$. set $e = x/\sqrt{2}$, $\exists\, e, e \in \mathbb{Q}^c$, $0<e<x$

$(0,1] \longrightarrow$ infinite irrational numbers, rational numbers

EXAMPLE 4. *Show that $\sqrt{2} + \sqrt{5}$ is irrational.*

Assume $\sqrt{2}+\sqrt{5}$ is rational. Then $\sqrt{2}+\sqrt{5} = p/q = x$, $x \in \mathbb{Q}$

$[(a+b)^2 = a^2+b^2+2ab]$

$\Rightarrow 7 + 2\sqrt{10} = p^2/q^2$

$\Rightarrow 2\sqrt{10} = p^2/q^2 - 7$

(Contradiction) $\Rightarrow \sqrt{10} = \boxed{\frac{1}{2}\left(p^2/q^2 - 7\right)}$

① $0.454545\cdots \Rightarrow p/q$ form

$$\left.\begin{aligned} x &= 0.454545\cdots \\ 100x &= 45.454545\cdots \end{aligned}\right\} \quad 99x = 45 \Rightarrow x = 45/99 = 5/11$$

② $0.761761761\cdots$ $\quad x = 761/999$

③ $0.142871428714287\cdots$ $\quad x = \frac{14287}{99999}$

④ $0.12345454545\cdots$

$10^3 x = 123.4545\cdots$

$10^5 x = 12345.4545\cdots$

$(10^5 - 10^3)x = 12222$

$x = \frac{\cancel{12222}}{\cancel{99000}} \quad \frac{1358}{11000}$

$= \boxed{\frac{679}{5500}}$

Equations

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(1)
$$x - y = 5$$
$$2x - 2y = 10$$
$$x - y = 5$$

Let $y = k$, $x = k + 5$

Solution set $= \{(k+5, k), k \in \mathbb{R}\}$

(2)
$$2x + 3y = 7$$
$$2 \times (x + 2y = 3)$$
$$2x + 4y = 6$$
$$y = -1$$
$$x - 2 = 3$$
$$\Rightarrow x = 5$$

$I_{2\times 2}$

MATRIX

$$\begin{array}{cccc} x & y & & \text{constant} \\ \end{array}$$
$$\checkmark \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & - \\ 0 & 1 & - \end{array}\right] \begin{array}{l} \rightarrow x \\ \rightarrow y \end{array}$$

Echelon form

Gaussian Elimination

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 2 & 3 \end{array}\right] \begin{array}{l} - R_1 \\ - R_2 \end{array}$$

$R_1 = R_1 - R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 2 & 3 \end{array}\right]$$

$R_2 = R_2 - R_1$
$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

$R_1 = R_1 - R_2$
$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right] \begin{array}{l} \rightarrow x \\ \rightarrow y \end{array}$$

(10)
$$x + y + z = 1$$
$$y + z = -1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 = R_1 - R_2$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow$$

$z = k$, $x = 2$

$y + k = -1 \Rightarrow y = -1 - k$

Solution set $= \{(2, (-1-k), k), k \in \mathbb{R}\}$

$$3x + y + z = 1$$
$$6x - y = 7$$
$$6y + z = 5$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 6 & -1 & 0 & 7 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -5 & 0 & -4 \\ 6 & -1 & 0 & 7 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

$R_1 = R_1 - R_3$

$$\left[\begin{array}{ccc|c} 27 & 0 & 0 & -39 \\ 6 & -1 & 0 & 7 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

$R_1 = R_1 - 5R_2$

$$\left[\begin{array}{ccc|c} 27 & 0 & 0 & -39 \\ 0 & -1 & 0 & 7 + \frac{2}{9} \cdot 39 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

$R_2 = R_2 - \frac{2}{9} R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -39/27 \\ 0 & 1 & 0 & -47/3 \\ 0 & 6 & 1 & 5 \end{array}\right]$$

$R_1 = R_1/27$

$R_2 = -R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -13/9 \\ 0 & 1 & 0 & -47/3 \\ 0 & 0 & 1 & 5 + 6 \cdot 47/3 \end{array}\right]$$

$R_3 = R_3 - 6R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -13/9 \\ 0 & 1 & 0 & -47/3 \\ 0 & 0 & 1 & 99 \end{array}\right] \begin{array}{l} \rightarrow x \\ \rightarrow y \\ \rightarrow z \end{array}$$

Equations

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✓ $x+y+z=1$
$3x+3y+z=5$

$$\left[\begin{array}{ccc|c} 3 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - 3R_2 \qquad \left[\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 = R_2 - R_1$
$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to z=-1 \\ \to y \\ \to x \end{array}$$

let $x=k$
$k+y=2$
$\Rightarrow y = 2-k$

solution set $= \{(k, 2-k, -1), k \in \mathbb{R}\}$

(12) $x+y=9$
$2x+y=3$
$x-y=4$

✓ $$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 0 & 13 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 0 & 4 \end{array}\right] \quad R_1 = R_1 + R_3$$

✓ $$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 13 \\ 0 & 1 & 0 & -10 \\ 1 & -1 & 0 & 4 \end{array}\right] \quad R_2 = R_2 - R_1 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 13/2 \\ 0 & 1 & 0 & -10 \\ 1 & 0 & 0 & -6 \end{array}\right] \quad R_3 = R_3 + R_2$$

Inconsistent system / no solution

Best / 2nd best

$x = 13/2, \; y = -10, \; x = -6$ ✗

HW

$$x+y-z=2$$
$$-x+y+z=4$$
$$x-y+z=6$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ -1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 6 \end{array}\right] \quad R_1 = \frac{R_1 + R_3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ -1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 6 \end{array}\right] \quad R_2 = R_1 + R_2 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 8 \\ 1 & -1 & 1 & 6 \end{array}\right] \quad R_3 = R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 8 \\ 1 & 0 & 2 & 14 \end{array}\right] \quad R_3 = \frac{R_3 - R_1}{2} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad R_2 = R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Alt

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ -1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 6 \end{array}\right] \quad R_2 = \frac{R_2 + R_3}{2} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 1 & -1 & 1 & 6 \end{array}\right] \begin{array}{l} \to x \\ \to z \\ \end{array}$$

$R_3 = (R_3 - R_1 - R_2)(-1)$

Equations

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$$x - y + w + z = 10 \quad — (1)$$
$$y - z = 4$$
$$x + w = 14$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 10 \\ 0 & 1 & -1 & 0 & 4 \\ 1 & 0 & 0 & 1 & 14 \end{array}\right] \quad R_1 = R_1 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 14 \\ 0 & 1 & -1 & 0 & 4 \\ 1 & 0 & 0 & 1 & 14 \end{array}\right] \quad R_3 = R_3 - R_1$$

(m, n)

$$— \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 14 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z = m$, $x = 14 - m$

$w = n$, $y = 4 + w$

solution $= \{(14-m, 4+n, m, n),\ m, n \in \mathbb{R}\}$

Determinant (sq matrix)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) \text{ or } |A| = ad - bc$$

$$A_1 = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \qquad |A| = bc - ad$$

$$\begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$$

$$\begin{cases} ax + by = c \\ a_1 x + b_1 y = c_1 \end{cases}$$

(1) unique solution exists if $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \neq 0$

(2) a. If $\det = 0$, & $\dfrac{a}{a_1} = \dfrac{b}{b_1} \neq \dfrac{c}{c_1}$ (Inconsistent / parallel lines)

b. If $\det = 0$, & $\dfrac{a}{a_1} = \dfrac{b}{b_1} = \dfrac{c}{c_1}$ (Infinite / coincident lines)

$$A = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \qquad a\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} + b\begin{bmatrix} c_1 & a_1 \\ c_2 & a_2 \end{bmatrix} + c\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

(Laplace Expansion)

$$= a(b_1c_2 - b_2c_1) + b(c_1a_2 - a_1c_2) + c(a_1b_2 - b_1a_2)$$

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 3 & 0 & -1 \\ 1 & -1 & 4 \end{bmatrix} \qquad 1(0-1) + (-2)(-1-12) + (-3)(-3-0)$$

$$= -1 + 26 + 9 = 34$$

$$A = \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a & b & c \end{bmatrix} \qquad a(b_1c - bc_1) + b(c_1a - ca_1) + c(a_1b - b_1a)$$

$$= \cancel{ab_1c} - \cancel{abc_1} + \cancel{bc_1a} - \cancel{bca_1} + \cancel{ca_1b} - \cancel{cb_1a} = 0$$

Inequalities

Sunday, April 23, 2023 9:00 AM

Arithmetic mean $= \underline{AM} = \dfrac{a+b}{2}$

Geometric mean $= \underline{GM} = \sqrt{ab}$

Harmonic mean $= \underline{HM} = \dfrac{2ab}{a+b}$

$(AM \geq GM \geq HM)$

a c b

$\dfrac{a}{c} = \dfrac{c}{b}$

$\dfrac{\frac{1}{a}+\frac{1}{b}}{2}$

$\dfrac{2}{1/a + 1/b}$

Check $a=b$

Cauchy-Schwarz Inequality

Sigma

DEFINITION

Cauchy-Schwarz inequality states that for all real numbers a_i and b_i, we have

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2,$$

where equality holds if and only if $\frac{a_i}{b_i} = k$ for some constant $k \in \mathbb{R}^+$, for all $1 \leq i \leq n$ which have $a_i b_i \neq 0$.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

Take $n=2$ $(a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2$

$(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$

(Fibonacci - Brahmagupta Identity)

Suppose we want to find the minimum value of the expression x² + 4y², subject to the condition that x + y = 5.

$1, 2, x, y \longrightarrow (1^2+2^2)(x^2+y^2) \geq (x+2y)^2 \Rightarrow 5(x^2+y^2) \geq (x+2y)^2$

Use $x = 5-y$, $5\left[(5-y)^2 + y^2\right] \geq (5-y+2y)^2$

$\Rightarrow 5\left[25 + 2y^2 - 10y\right] \geq (5+y)^2$

$\Rightarrow 125 + 10y^2 - 50y \geq 25 + y^2 + 10y$

$y \geq 0$

$x^2 + 4y^2 \geq 0$ ✓

$(x+y=5)$

$\begin{cases} y=0, x=5 \\ x=0, y=5 \end{cases}$

(a + b + c)(1/a + 1/b + 1/c) ≥ 9 , $a, b, c \in \mathbb{R}^+$

$\sqrt{a}, \sqrt{b}, \sqrt{c}$, $\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}$

$$= \left[(\sqrt{a})^2 + (\sqrt{b})^2 + (\sqrt{c})^2\right]\left[\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2\right]$$

$$\geq \left(\sqrt{a}\cdot\frac{1}{\sqrt{a}} + \sqrt{b}\cdot\frac{1}{\sqrt{b}} + \sqrt{c}\cdot\frac{1}{\sqrt{c}}\right)^2$$

Inequalities

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$$\left\{ \prod a_i = 1, \quad a_i \in \mathbb{R}^+ \right.$$

(Nesbitt's Inequality)

$$\left(a_1 + \frac{1}{a_2}\right)\left(a_2 + \frac{1}{a_3}\right) \cdots \left(a_{n-1} + \frac{1}{a_n}\right)\left(a_n + \frac{1}{a_1}\right) \geq 2^n$$

LHS

$$\frac{(a_1a_2+1)(a_2a_3+1)\cdots(a_na_1+1)}{\prod a_i = 1}$$

$$(a_1a_2+1)(a_2a_3+1)\cdots(a_na_1+1) \geq 2^n\sqrt{a_1a_2\cdots a_n}$$

$$= 2^n a_1a_2a_3 \cdots \sqrt{a_1a_2a_3}$$

$$= 2^n \cdot \prod a_i$$

(from AM-GM inequality)

$$\frac{a_1a_2+1}{2} \geq \sqrt{a_1a_2 \cdot 1}$$

$$a_1a_2+1 \geq 2\sqrt{a_1a_2}$$

$$a_2a_3+1 \geq 2\sqrt{a_2a_3}$$

a_2

$$\frac{(a_1a_2+1)(a_2a_3+1)\cdots(a_na_1+1)}{\prod a_i} \geq 2^n$$

Proved

let $a, b \in \mathbb{R}^+$,

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{a^2+b^2} \geq \frac{32(a^2+b^2)}{(a+b)^4}$$

1st term $\rightarrow$ $\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$ x

2nd term $\rightarrow$ $\frac{4}{a^2+b^2}$ y

Use AM $\geq$ GM

$$\frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{4}{a^2+b^2}\right) \geq \sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \cdot \frac{4}{a^2+b^2}}$$

$$\Rightarrow \text{LHS} \geq 2\sqrt{\frac{\cancel{a^2+b^2}}{a^2b^2} \cdot \frac{4}{\cancel{a^2+b^2}}}$$

$$= \frac{4}{ab}$$

$$\left\{ \frac{4}{ab} \geq \frac{32(a^2+b^2)}{(a+b)^4} \right.$$ if true

This inequality is equivalent to $\rightarrow$

$$\text{LHS} \geq \frac{4}{ab} \geq \text{RHS}$$ ✓

$$\frac{4}{ab} \geq \frac{32(a^2+b^2)}{(a+b)^4}$$ ✓

$$\Rightarrow 4(a+b)^4 \geq 32(a^2+b^2)ab$$

$$\Rightarrow (a+b)^4 - 8ab(a^2+b^2) \geq 0$$

$$\Rightarrow a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \geq 0$$ ✓

Inequalities

Sunday, April 30, 2023 9:00 AM

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}, n \in \mathbb{N}$$

$$b^{n+1} - a^{n+1} = (b-a)(b^n + b^{n-1}a + \cdots + a^n)$$

(for $b>a$)

$$\rightarrow (n+1)a^n < b^n + b^{n-1}a + b^{n-2}a^2 + \cdots + a^n < (n+1)b^n$$

$$= (n+1)a^n(b-a) < b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \quad (1)$$ (multiplying by $b-a \neq 0$ throughout)

Use $a = \left(1+\frac{1}{n+1}\right)$, $b = \left(1+\frac{1}{n}\right)$

Then $b-a = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

Replace the values in (1)

$$\therefore \frac{1}{n}\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

Consider, $\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

$$\Rightarrow \left(1+\frac{1}{n}\right)^{n+1} - \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n \left\{1+\frac{1}{n}-\frac{1}{n}\right\} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

Consider a^2, b^2, c^2

$$\frac{a^2+b^2}{2} \geq \sqrt{a^2 \cdot b^2}$$

$$\Rightarrow \frac{a^2+b^2}{2} \geq ab$$

(Add it cyclically)

If $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$, and each of them is less than 1, $s_n = \sum a_i$, then

(i) $1 - s_n < (1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1-s_n}$

(ii) $1 + s_n < (1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{1+s_n}$

Weierstrass Inequality

Observe $(1-a_1)(1-a_2) = 1 - a_1 - a_2 + a_1a_2 > 1-(a_1+a_2)$

and $(1-a_1)(1-a_2)(1-a_3) = 1-(a_1+a_2) - a_3 + (a_1+a_2)a_3$

$$> 1 - (a_1+a_2+a_3)$$

Using induction, $\prod(1-a_i) > 1 - \sum a_i = 1 - s_n$ (definition of s_n)

Similarly, we can show $\prod(1+a_i) > 1 + s_n$

We know $(1+a_i)(1-a_i) < 1$, as $a_i < 1$ and $a_i \in \mathbb{R}^+$

$$\Rightarrow (1-a_i) < \frac{1}{(1+a_i)} \quad \text{for } i \in \mathbb{N}$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+a_1}\cdot\frac{1}{1+a_2}\cdots\frac{1}{1+a_n} < \frac{1}{1+s_n}$$

$$\Rightarrow \prod(1-a_i) < \frac{1}{\prod(1+a_i)} < \frac{1}{1+s_n}$$

Inequalities

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If $a_i \in \mathbb{R}^+$ and $a_i \geq 1$, then show

$$\prod(1+a_i) \geq (1+\sum a_i)\cdot\frac{2^n}{1+n}$$

Observe $1+a_i = 2\left(\frac{1}{2}+\frac{a_i}{2}\right)$, extending:

(multiplying)

$$\prod(1+a_i) = 2^n \prod\left(\frac{1}{2}+\frac{a_i}{2}\right)$$

$$= 2^n \prod\left(1+\frac{a_i-1}{2}\right)$$

$$\geq 2^n\left(1+\frac{a_1-1}{2}+\frac{a_2-1}{2}+\cdots+\frac{a_n-1}{2}\right)$$

[Using $\prod(1+a_i) \geq 1+\sum a_i$]

$$\geq 2^n\left(1+\frac{a_1-1}{n+1}+\frac{a_2-1}{n+1}+\cdots+\frac{a_n-1}{n+1}\right)$$

$$= \frac{2^n}{n+1}(1+a_1+a_2\cdots+a_n)$$

For any n, $\frac{1}{2\sqrt{n+1}} < \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

Inequalities

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(1) $a > b,\ b > c$, then $a > c$

(2) $a > b$, then $a + c > b + c$, $c \geq 0$

(3) $a + b > c + d \Rightarrow a + b - c > d$ (can transpose signs)

(4) $a > b \Rightarrow b < a$

(5) $a > b \Rightarrow ac > bc$ & $a/c > b/c$, for $c > 0$

(6) $a > b \Rightarrow -b > -a$

(7) If $a > b$, then $1/a < 1/b$ for $a, b > 0$

(8) $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n \Rightarrow \sum^n a_i > \sum^n b_i$

(9) $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n \Rightarrow \prod a_i > \prod b_i$

} positive numbers

(10) $a > b$, $a^n > b^n$ and $a^{1/n} > b^{1/n}$

→ Which is one is greater? $(31)^{12}$ or $(17)^{17}$

$31 < 32$, $(31)^{12} < (32)^{12} = (2^5)^{12} = 2^{60} < 2^{68}$ $[(a^m)^n = a^{mn}]$

$\hookrightarrow = 2^{4 \times 17} = (2^4)^{17} = (16)^{17}$

$\therefore (31)^{12} < 2^{60} < 2^{68} = 16^{17} < 17^{17}$

We know, $17 > 16$, $(17)^{17} > (16)^{17}$. Hence, $(17)^{17} > (31)^{12}$

→ Which is greater? $(30)^{100}$ or $\underline{2^{567}}$

$30 < 32 \Rightarrow 30^{100} < 32^{100} = (2^{10})^{50} = (1024)^{50} < (1024)^{54}$

$= (2^{10})^{54} = (2^{20})^{27}$

$(2^{20})^{27} < (2^{21})^{27} = 2^{567}$

$\therefore 30^{100} < 2^{567}$

HW 1: 7^{92} or 8^{91} ?

HW 2: 150^{300} or $(20000)^{100} \times (100)^{100}$

→ Show $(1.01)^{1000} > 1000$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

0 0 1 0 0

0 1 1 0

0 1 2 1 0

0 1 3 3 1 0

0 1 4 6 4 1 0

1 5 10 10 5 1

1 6 15 20 15 6 1

} Pascal's triangle

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_n b^n$ → Binomial theorem

$= 1.a^n + C_1 a^{n-1} b + C_2 a^{n-2} b^2 + \cdots + C_n b^n$

Put $a = 1, b = x$: $(1+x)^n = 1 + C_1 x + \underbrace{C_2 x^2 + \cdots + C_n x^n}$ positive, if $x > 0$

$\Rightarrow (1+x)^n > 1 + C_1 x = 1 + nx$ $\therefore$ $\boxed{(1+x)^n > 1 + nx}$

Inequalities

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$$LHS > (1\cdot4)^{25} > (1\cdot4)^{24}$$
$$= (1\cdot4^3)^8$$
$$> (2\cdot7)^8$$
$$> (2\cdot5)^8$$
$$> 5^{8/2} > 1000$$

$$(1+0\cdot01)^8 > 1+8\times0\cdot01 = 1\cdot08$$
$$[(1+0\cdot01)^8]^{125} > (1+0\cdot08)^{125}$$
$$= [(1+0\cdot08)^5]^{25}$$
$$> (1+0\cdot4)^{25}$$

Which one is bigger,

$$\boxed{(150)^{300}} \text{ or } (20000)^{100}\times(100)^{100}$$

$$(150^3)^{100}$$
$$= [(30\times5)^3]^{100}$$
$$= (30^3)^{100}\times(5^3)^{100}$$
$$= (27000)^{100}\times(125)^{100}$$

Show that

$$\sqrt2 < \frac{\sqrt2+\sqrt3}{\sqrt{2+\sqrt3}} < 2$$

$\sqrt3>\sqrt2$ and $\sqrt3<2$ (known)

$$\sqrt3>\sqrt2 \Rightarrow \sqrt2+\sqrt3 > 2\sqrt2 \quad \text{--- (1)}$$

$$\sqrt3<2 \Rightarrow 2+2>\sqrt3+2$$
$$\Rightarrow \sqrt4 > \sqrt{\sqrt3+2}$$
$$\Rightarrow 2 > \sqrt{2+\sqrt3} \Rightarrow \frac{1}{\sqrt{2+\sqrt3}} < \frac12 \quad \text{--- (2)}$$

Multiplying (1) & (2), $\dfrac{\sqrt2+\sqrt3}{\sqrt{2+\sqrt3}} > \sqrt2$ --- (3)

Again, $\sqrt2<\sqrt3$, $\sqrt2+\sqrt3 < 2\sqrt3$ --- (4)

$$1<\sqrt3 \Rightarrow 1+2<\sqrt3+2 \Rightarrow \sqrt3 < \sqrt{2+\sqrt3}$$
$$\Rightarrow \frac{1}{\sqrt{2+\sqrt3}} < \frac{1}{\sqrt3} \quad \text{--- (5)}$$

Multiplying (4) & (5)

$$\frac{\sqrt2+\sqrt3}{\sqrt{2+\sqrt3}} < 2\cancel{\sqrt3}\cdot\frac{1}{\cancel{\sqrt3}} \quad \text{--- (6)}$$

Combine (3) & (6) $\rightarrow$ $\sqrt2 < \dfrac{\sqrt2+\sqrt3}{\sqrt{2+\sqrt3}} < 2$ (proved)

$$\frac{101}{1001} > \frac{10001}{100001} > \frac{1000001}{10000001}$$

$101 \rightarrow 10x \rightarrow 1010 = 10x-9$

$D = 10N-9$

$$\frac{x}{10x-9} = a$$

then $\dfrac1a = \dfrac{10x-9}{x} = 10 - 9/x$

If x increases, $9/x$ decreases and $1/a$ increases $\Rightarrow$ a decreases

$f(x), g(x)$, $\rightarrow$ check $f(x)-g(x)$

Show $m^3+1 > m^2+m$, for $m>-1$, $m\neq1$, $m\in\mathbb{Z}$ (try with difference)

Inequalities

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$(m^3+1)-(m^2+m) = (m^3-m^2)-(m-1)$

$= m^2(m-1)-1(m-1)$

$= (m-1)(m^2-1) = (m-1)^2(m+1)$

$(m-1)^2(m+1)$ is always positive for $m>-1,\ m\neq 1,\ m\in\mathbb{Z}$

$(m^3+1)-(m^2+m) > 0$

$\Rightarrow (m^3+1) > (m^2+m)$

$[m^3+1 = m^2+m, \text{ if } m=-1]$

HW

1. $a^3b+ab^3 < a^4+b^4,\ a\neq b$
2. $x^5+y^5 > x^4y+xy^4,\ x\neq y$

If $a>b$, and x is positive, then show $\frac{a+x}{b+x} < \frac{a}{b}$

$$\frac{a+x}{b+x}-\frac{a}{b} = \frac{b(a+x)-a(b+x)}{b(b+x)} = \frac{-x(a-b)}{b(b+x)} < 0$$

Given, $a>b \Rightarrow (a-b)>0$, also, $x>0$

If $x>0$, then show $\left(x+\frac{1}{x}\right) \geq 2$

$$x+\frac{1}{x}-2 = \frac{x^2+1-2x}{x} = \begin{cases} \frac{(x-1)^2}{x} > 0, & \text{if } x\neq 1 \\ 0, & \text{if } x=1 \end{cases}$$

$x+\frac{1}{x} > 2$ if $x\neq 1$

and $x+\frac{1}{x} = 2$ if $x=1$

Combining, $x+\frac{1}{x} \geq 2$ if $x>0$

[If $x<0$, then $x+\frac{1}{x} \leq -2$]

Inequalities

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