The texts "Numbers and Proof", by R B J T Allenby, Edward Arnold, 1997, Elementary Number Theory", by GA & J M Jones, Springer Undergraduate Mathematics Series, 1998

"Elements of Logic via Sets and Numbers" by D L Johnson and "Introductory Mathematics: Algebra and Analysis" by G Smith, both in the Springer Undergraduate Mathematics Series

"A Concise Introduction to Pure Mathematics" by Martin Liebeck and "How to Think Like a Mathematician: A Companion to Undergraduate Mathematics" by Kevin Houston

Lara Alcock's book "How to Study for a Mathematics Degree"

"What is Mathematics?", by R Courant and H Robbins, Oxford University Press, 1996

This question is about *influencer networks*. An influencer network consists of n *influencers* denoted by circles, and arrows between them. Throughout this question, each influencer holds one of two opinions, represented by either a $\triangle$ or a $\square$ in the circle. We say that an influencer A follows influencer B if there is an arrow from B to A; this indicates that B has ability to influence A.

Logic

Algorithm

Binary.

n dimensional

$\boxed{m \times n}$ matrix

Binary. 2^n

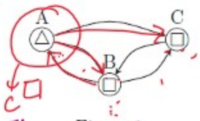


Figure 1

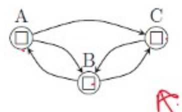


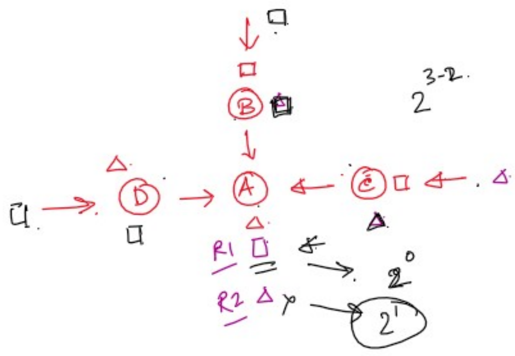
Figure 2

$B \to A.$

A follows B

stable

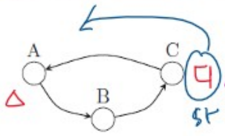
% of influencers



The example in Figure 1 above shows a network with A and B following each other, B and C following each other, and C also following A. In this example, initially B and C have opinion $\square$, while A has opinion $\triangle$. An influencer will change their mind according to the *strict* majority rule, that is, they change their opinion if *strictly* more than half of the influencers they are following have an opinion different from theirs. Opinions in an influence network change in rounds. In each round, each influencer will look at the influencers they are following and *simultaneously* change their opinion at the end of the round according to the strict majority rule. In the above network, after one round, A changes their opinion because the only influencer they are following, B, has a differing opinion, and the network becomes as shown in Figure 2 above.

An influencer network with an initial set of opinions is *stable* if no influencer changes their opinion, and a network (with initial opinions) is *eventually stable* if after a finite number of rounds it becomes stable. The network in the above example is eventually stable as it becomes stable after one round.

(i) A network of three influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers A, B and C? Justify your answer.



NO

St R1 R2 R3

C1 C2 C3 C4

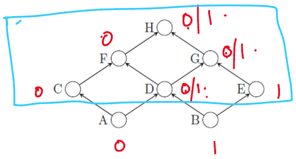
If all 3 have the same opinions at the start then it will be stable else no.

2^1

(ii) Another network of influencers (without opinions) is shown below. Is this influencer network eventually stable regardless of the initial opinions of the influencers? Justify your answer.

$\triangle \to 0$

encer network eventually stable regardless of the initial opinions of the influencers? Justify your answer.



Δ → 0

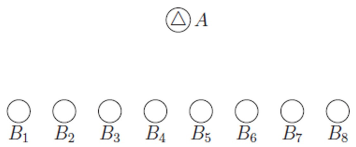
□ → 1

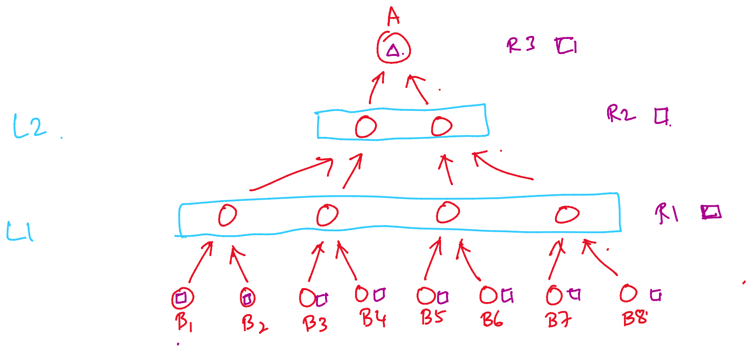
Case 1 A, B → 0.

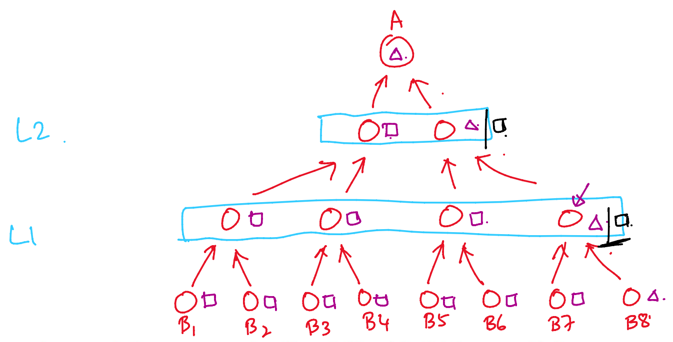
Case 2 B/A = 0 A/B = 1

Case 3 A, B → 1

(iii) A *partial* network of influencers (without opinions for $B_1, \ldots, B_8$) is shown below. You can add at most six additional influencers, assign any opinion of your choice to the new influencers, and add any arrows to the network to describe follower relationships. Design a network that is eventually stable regardless of initial opinions, and has the property that when it becomes *stable* A has opinion □ if and only if each of $B_1, B_2, \ldots, B_8$ had opinion □ at the start. Justify your answer.

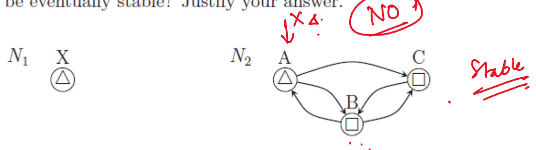




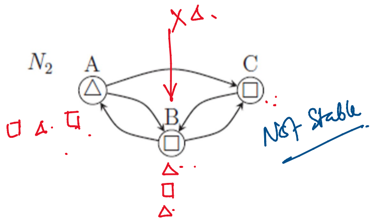


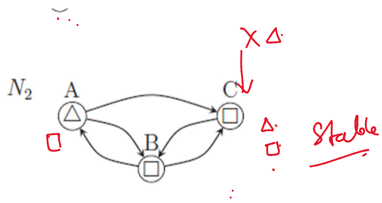
(iv) You are given two influencer networks, N_1 and N_2, with disjoint sets of influencers shown below. Both are *eventually stable*. Suppose one of the influencers from network N_2 follows the influencer X from the network N_1. Is the resulting network guaranteed to be eventually stable? Justify your answer.

NO



Stable





(v) (a) Given a network with n influencers, where the arrows are fixed, but you are allowed to assign *opinions* ($\triangle$ or $\square$) to each influencer, how many possible assignments of opinions is possible?

(b) Given an influencer network and an initial assignment of opinions, explain how you would determine whether the influencer network is eventually stable. Justify your answer.

Distinct numbers are arranged in an $m \times n$ rectangular table with m *rows* and n *columns* so that in each row the numbers are in increasing order (left to right), and in each column the numbers are in increasing order (top to bottom). Such a table is called a *sorted* table and each location of the table containing a number is called a *cell*. Two examples of sorted tables with 3 rows and 4 columns (and thus $3 \times 4 = 12$ cells) are shown below.

3	12	33	64
15	26	37	78
19	40	51	92

5	22	53	68
18	36	67	78
19	45	81	92

We index the cells of the table with a pair of integers (i, j), with the top-left corner being $(1, 1)$ and the bottom-right corner being (m, n). Observe that the smallest entry in a sorted table can only occur in cell $(1, 1)$; however, note that the second smallest entry can appear either in cell $(1, 2)$, as in the first example above, or in cell $(2, 1)$ as in the second example above.

(i) (a) Assuming that $m, n \geqslant 3$, where in an $m \times n$ sorted table can the third-smallest entry appear?

(b) For any $k \geqslant 4$ satisfying $m, n \geqslant k$, where in an $m \times n$ sorted table can the k^{th} smallest entry appear? Justify your answer.

(ii) Given an $m \times n$ sorted table, consider the problem of determining whether a particular number y appears in the table. Outline a procedure that inspects at most $m + n - 1$ cells in the table, and that correctly determines whether or not y appears in the table. Briefly justify why your procedure terminates correctly in no more than $m + n - 1$ steps.
[Hint: As the first step, consider inspecting the top-right cell.]

(iii) Consider an $m \times n$ table, say A, which might not be sorted; an example is shown below. Obtain table B from A by re-arranging the entries in each row so that they are in sorted order. Then obtain table C from B by re-arranging the entries in each column so that they are in sorted order. Fill in tables B and C here:

A:

33	92	46	24
25	26	37	8
49	40	81	22

$\rightarrow$

B:

$\rightarrow$

C:

(iv) Show that for *any* $m \times n$ table A, performing the two operations from part (iii) results in a sorted table C.