Differential calculus (Revision)

Q. $\lim_{x \to \infty} \left(\frac{x+6}{x+1} \right)^{x+4} = ?$

(a) 0 (c) e^5

(b) 1 (d) e^3

$\lim_{x \to \infty} \left(\frac{x+6}{x+1} \right)^{x+4}$ (1^∞)

L. Hopital's Rule: $\left(\frac{0}{0}\right), \left(\frac{\infty}{\infty}\right)$

Note:

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$

$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{x} = e^a$

$y = \left(\frac{x+6}{x+1} \right)^{x+4}$

$\log y = (x+4) \log \left(\frac{x+6}{x+1} \right)$

$= \frac{\log \left(\frac{x+6}{x+1} \right)}{1/(x+4)}$ $\left[\frac{0}{0}\right]$

use- L. Hopital's Rule.

Alternative: $\lim_{x \to \infty} \left(\frac{x+6}{x+1} \right)^{x+4} = \lim_{x \to \infty} \left(1 + \frac{5}{x+1} \right)^{x+4}$

$= \lim_{x \to \infty} \underbrace{\left(1 + \frac{5}{x+1} \right)^{x+1}}_{e^5} \left(1 + \frac{5}{x+1} \right)^3.$

$= e^5 . \lim_{x \to \infty} \left(1 + \frac{5}{x+1} \right)^3 = e^5.$

Q. Evaluate: $\lim_{x \to 0}. \left[\frac{\log_e \left(\sum_{k=0}^{n} x^k \right)}{\sum_{k=1}^{\infty} \frac{x^k}{k!}} \right], n \geq 2$

(a) 1 (c) -1

(b) 0 (d) None

Deno. $\sum_{k=1}^{\infty} \frac{x^k}{k!} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

$= (e^x - 1) \quad \left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \infty \right]$

Num: $\log_e \left(\sum_{k=0}^{n} x^k \right) = \log_e \left(1 + x + x^2 + \cdots + x^n \right)$.

$= \log_e \left(\frac{1 - x^{n+1}}{1 - x} \right)$ [∵ We'll evaluate at $\lim_{x \to 0}$]

$= \ln \left(1 - x^{n+1} \right) - \ln (1 - x)$

$\therefore \lim_{x \to 0} \frac{\ln(1 - x^{n+1}) - \ln(1-x)}{e^x - 1} \quad \left[\frac{0}{0} \right]$

L·Hopital: $\lim_{x \to 0} \frac{\frac{1}{1 - x^{n+1}} - (n+1) x^n - \frac{1}{1-x}(-1)}{e^x}$

$= 1$.

Q. Let $f(x) = x|x|e^{-x}$.

(a) $f(x)$ is continuous but not diff at $x = 0$.

(b) Diff at $x = 0$

(c) Diff everywhere.

(d) Diff at finitely many points.

Product of 2 cont fn $\Rightarrow f(x)$ is continuous.

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \qquad \therefore f(x) = \begin{cases} x^2 e^{-x}, & x \geq 0 \\ -x^2 e^{-x}, & x < 0 \end{cases}$

To check: LHD: $\lim\limits_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h}$ RHD: $\lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h}$.

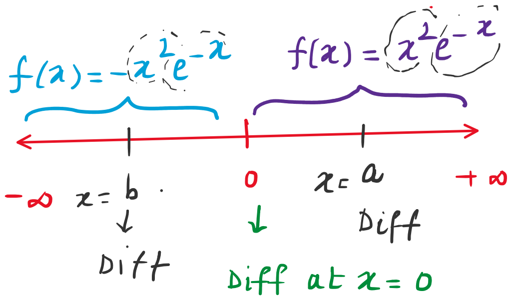
$f(0)=0$. : LHD $= \lim\limits_{h\to 0^-} -\dfrac{h^2 e^{-h}}{h} = -\lim\limits_{h\to 0} h e^{-h} = 0$.

RHD $= \lim\limits_{x\to 0^+} \dfrac{h^2 e^{-h}}{h} = \lim\limits_{h\to 0} h e^{-h} = 0$.

LHD = RHD at $x=0$ $\Rightarrow$ $f(x)$ is differential at $x=0$.

Let $a>0$. $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

Diff at $x=a$.



Let $b<0$, $\lim\limits_{h\to 0} \dfrac{f(b+h)-f(b)}{h}$

Diff at $x=b$.

$\Rightarrow$ $f(x)$ is diff everywhere. (c)

Q. Let $f(x) = \begin{cases} \dfrac{\sin x^2}{x}, & x\neq 0. \\ 0, & x=0. \end{cases}$

(a) Cont but not diff at $x=0$.

(b) Diff at $x=0$ but derivative is not cont at $x=0$

(c) ✓ Diff at $x=0$ & derivative is cont at $x=0$

(d) Not cont at $x=0$.

Check for diff at $x=0$: $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim \dfrac{\sin h^2}{\ } = 1$

Check for diff at $x=0$ :: $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{\sin h^2}{h^2} = 1$

$$f(x) = \frac{\sin x^2}{x}$$

$$f'(x) = 2\cos x^2 - \frac{\sin x^2}{x^2} \qquad f'(0) = 1.$$

Check cont. of $f'(x)$ at $x=0$.

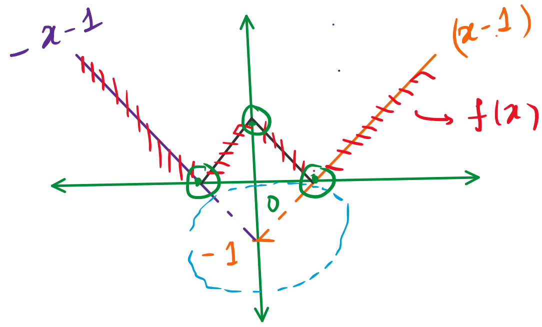
RHL: $\lim_{h\to 0} f'(0+h) = \lim_{h\to 0} f'(h) = \lim_{h\to 0} 2\cos h^2 - \frac{\sin h^2}{h^2}$

$= 2-1 = 1 = f'(0)$

$f'(x)$ continuous at $x=0$.

Q. Let $f(x) = \big|\,|x|-1\,\big|$. The number of points at which $f(x)$ is not differentiable is: (a) 1 (b) 2 (c) 3 (d) None.

$$|x|-1 = \begin{cases} -x-1, & x<0 \\ x-1, & x\geq 0. \end{cases}$$



HW

Q. $f(x) = \begin{cases} \big|\,|x-1|-1\,\big|, & x<1 \\ [x], & x\geq 1 \end{cases}$

Then the points of discontinuity of $f(x)$ is:

(a) all integers ≥ 0 (c) All integer > 1

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(b) all integers ≥ 1

(c) All integer > 1

(d) only $x = 1$.