1. Geometric Mean (G.M)

(a) without frequency

$$GM = \left(x_1\, x_2 \cdots x_n\right)^{1/n}$$

$$GM = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

(b) with frequency $(f_1\, f_2 \cdots f_n)$

$$GM = \left(x_1^{f_1}\, x_2^{f_2} \cdots x_n^{f_n}\right)^{1/N}$$

where $N = \sum_{i=1}^{n} f_i$

or, $$GM = \left(\prod_{i=1}^{n} x_i^{f_i}\right)^{1/N}$$

(i) if all observations are same, then GM is the same value.

ie let $x_i = c$ for all $i = 1\,(1)\,n$.

then $GM = (c \cdot c \cdot c \cdots c)^{1/n}$

$GM = (c^n)^{1/n} = c$

(Proved)

(ii) GM of (x/y) is the ratio of their GM

ie $GM(x/y) = GM(x)/GM(y)$

$$GM(x/y) = \left(\frac{x_1}{y_1}\, \frac{x_2}{y_2} \cdots \frac{x_n}{y_n}\right)^{1/n}$$

$GM(x/y) = \left(\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \cdots \frac{x_n}{y_n} \right)^{\cdots}$

$$= \frac{(x_1 \cdot x_2 \cdots x_n)^{1/n}}{(y_1 \cdot y_2 \cdots y_n)^{1/n}} = \frac{GM(x)}{GM(y)}$$

(Proved)

(iii) log of GM is the arithmetic mean of log of observation.

$GM(x) = (x_1 \, x_2 \ldots x_n)^{1/n}$

Taking log on both sides

$$\log(GM) = \frac{1}{n}\left[\log x_1 + \log x_2 + \cdots + \log x_n\right]$$

$$\log GM = \frac{1}{n}\sum_{i=1}^{n} \log(x_i)$$

(Proved)

Harmonic Mean

without frequency (a) $HM = \dfrac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$

$$\boxed{HM = \frac{n}{\sum_{i=1}^{n} 1/x_i}}$$

with frequency (b) $HM = \dfrac{\Sigma f_i}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \cdots + \frac{f_n}{x_n}}$

$x \quad \bar{x}_1 \quad \bar{x}_2 \quad \bar{x}_n$

$$\boxed{HM = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \frac{f_i}{x_i}}}$$

Properties

(1) $\boxed{AM \times HM = GM^2 \quad \text{or} \quad \sqrt{AM \times HM} = GM.}$

(2) $\boxed{AM \geq GM \geq HM}$

(1)

Let x_1 and x_2 be two observations

then $AM = A = \dfrac{x_1 + x_2}{2}$ — (1)

$GM = G = \sqrt{x_1 \cdot x_2}$ — (2)

$HM = H = \dfrac{2}{\frac{1}{x_1} + \frac{1}{x_2}}$ — (3)

Now $AM \times HM = \left(\dfrac{x_1 + x_2}{2}\right) \times \left(\dfrac{2}{\frac{1}{x_1} + \frac{1}{x_2}}\right)$

$= \left(\dfrac{\cancel{x_1 + x_2}}{\cancel{2}}\right) \times \dfrac{\cancel{2}}{\left(\frac{\cancel{x_1 + x_2}}{x_1 x_2}\right)}$

$= x_1 \cdot x_2$

$(\quad)^2$

$= x_1 x_2$

$= \left(\sqrt{x_1 x_2}\right)^2$

$AM \times HM = G^2 = GM^2$ —— (4)

or, $\boxed{\sqrt{AM \times HM} = GM}$ (Proved)

(2) $AM \geq GM \geq HM$.

Let us take n observations $x_1, x_2 \ldots x_n$

such that $AM = A = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ —— (1)

$GM = G = (x_1 x_2 \ldots x_n)^{1/n}$ —— (2)

$HM = H = \dfrac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$ —— (3)

Now let us take 2 obs x_1 and x_2

then $\left(\sqrt{x_1} - \sqrt{x_2}\right)^2 \geq 0$

or, $x_1 - 2\sqrt{x_1 x_2} + x_2 \geq 0$

or, $\dfrac{x_1 + x_2}{2} \geq \sqrt{x_1 x_2}$

$AM \geq GM$ for $n = 2$ obs.

Similarly for 2 more observations x_3 and x_4

Similarly for 2 more observations x_3 and x_4

$$\frac{x_3+x_4}{2} \geq \sqrt{x_3 x_4}$$

$$AM \geq GM$$

Now let us consider two quantities

$$\frac{x_1+x_2}{2} \quad \text{and} \quad \frac{x_3+x_4}{2}$$

$$\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2} \geq \sqrt{\left(\frac{x_1+x_2}{2}\right)\left(\frac{x_3+x_4}{2}\right)}$$

$$\frac{x_1+x_2+x_3+x_4}{4} \geq \sqrt{\sqrt{x_1 x_2}\sqrt{x_3 x_4}}$$

$$\frac{x_1+x_2+x_3+x_4}{4} \geq (x_1 x_2 x_3 x_4)^{1/4}$$

$AM \geq GM$ for $n=4$ obs

$\therefore$ Proceeding this way we can prove for $2, 4, 6, 8 \cdots 2^m$ observations where m is any +ve integer.

Let us take $2^{m-1} \leq n < 2^m$

...ition

Let us take $2 \leq n \leq 2$ such that $2^n = N$ (say) consisting of n observations $(x_1, x_2 \ldots x_n)$ and $(N-n)$ observations A where A is the arithmetic mean of 'n' observ $x_1, x_2 \ldots x_n$.

$n < N$

$N \to n + \underbrace{A \ldots A}_{N-n}$

$A =$ am of 'n'

$x_1, x_2 \ldots x_n$

$A = \frac{1}{n} \sum x_i$

i.e. $\left\{ \underbrace{x_1, x_2 \cdots x_n}_{n \text{ terms}} \; \underbrace{A \; A \ldots A}_{(N-n) \text{ term}} \right\} N$ terms.

$\therefore$ For N observation

$$AM = \frac{(x_1 + x_2 + \cdots + x_n) + (A + A + \cdots + A)}{N}$$

$$= \frac{nA + (N-n)A}{N}$$

$$= \frac{\cancel{nA} + \cancel{N}A - \cancel{nA}}{\cancel{N}}$$

$$\boxed{AM = A} \quad \text{--- (4)}$$

Again $GM = (\underbrace{x_1 x_2 \ldots x_n}\; \underbrace{A\,A \ldots A})^{1/N}$

$= \left(G^n \; A^{N-n} \right)^{1/N}$

$$= \left(G^n A^{N-n}\right)^{1/N}$$

for $N = 2^m$ obs, we know $AM \geq GM$

or $A \geq \left(G^n A^{N-n}\right)^{1/N}$

or, $A^N \geq G^n A^{N-n}$

or, $A^{N-N+n} \geq G^n$

or, $A^m \geq G^n$

or $\boxed{A \geq G}$ — (5)

for any no. of observations

Let us take the reciprocal of all 'n' observation

$\frac{1}{x_1}, \frac{1}{x_2} — \frac{1}{x_n}$

$$AM = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n}$$

$$\boxed{AM = \frac{1}{H}}$$

And $GM = \left(\frac{1}{x_1} \cdot \frac{1}{x_2} \cdots \frac{1}{x_n}\right)^{1/n} = \frac{1}{G}$

$$= \left(\frac{1}{G^n}\right)^{1/n}$$

Since $AM \geq GM$

$\Rightarrow \frac{1}{H} \geq \frac{1}{G}$

$\Rightarrow \boxed{G \geq H}$ — (6)

$$\Rightarrow \boxed{G \geq H} \longleftarrow (6)$$

Comparing (5) and (6), we have

$$\boxed{AM \geq GM \geq HM}$$

Proved

Median

$$\bar{x}_m = x_l + \left[\frac{N/2 - cf}{f_m}\right] \times c$$

x_l → lower boundary of median class

N → Total freq

cf ← cumm frequency preceding median class

f_m ← freq of median class

c ← class size

Mode $= x_l + \left[\frac{f_0 - f_{-1}}{2f_0 - f_{-1} - f_1}\right] \times c$

x_l ← lower limit of modal class

f_0 → frequency of modal class (Highest freq)

f_{-1} → frequency preceding

f_1 → frequency succeeding modal class.

⟨Q⟩

Values	300-325	325-350	350-375	375-400
Frequency	5	17	80	? (a)
	400-425	425-450	450-475	
	326	(b) ?	88	
			475-500	
			9	

It is known that total frequency is 1,000 and median is 413.11.

Find the missing frequency.

Class Boundary	Frequency	Cummulative Frequency
300-325	5	5
325-350	17	22
350-375	80	102
375-400	a	102+a (cf)
400-425	326 → f_m	428+a ← 500
425-450	b	428+a+b
450-475	88	516+a+b
475-500	9	525+a+b = 1000
	$\Sigma f = 1000$	

$$\Sigma f = 1000$$

Total frequency = 1000

$525 + a + b = 1000$

$a + b = 475$ —— (1)

Median = 413.11

$$x_l + \frac{N/2 - cf}{f_m} \times c = 413.11$$

$$400 + \left(\frac{500 - (102 + a)}{326}\right) \times 25 = 413.11$$

$(400 \times 326) + (500 - 102 - a) \times 25 = 413.11 \times 326$

$(130400) + (398 \times 25) - 25a = 134673.86$

$130400 + 9950 - 25a = 134673.86$

$\cancel{142850 - 25a}$

$140350 - 25a = 134673.86$

$25a = 140350 - 134673.\ldots$

$a = (227.048)$

(ans)

$a + b = 475$

$b = 475 - a = 475 - 227$

$b = 248$ (ans)