Discontinuous at integers

continuous.

Q. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as: $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$. Then:

(a) $f(x)$ is continuous $\forall x$

(b) discontinuous only at $x=0$

(c) discontinuous at all integers

(d) continuous only at $x=0$.

Let $n \in \mathbb{Z}$.

RHL: $\lim_{x \to n^+} [x] \cdot \cos\left(\frac{2x-1}{2}\right)\pi = n \cos\left(\frac{2n-1}{2}\right)\pi = 0$.

odd

LHL: $\lim_{x \to n^-} [x] \cdot \cos\left(\frac{2x-1}{2}\right)\pi = (n-1) \cos\left(\frac{2n-1}{2}\right)\pi = 0$.

Q. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ s.t $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in $[-1, 1]$:

(a) $P(-1)$ is not min, $P(1)$ is max

(b) $P(-1)$ is min, $P(1)$ is not max.

(c) Neither $P(-1)$, $P(1)$ are min & max

(d) $P(-1)$ is min, $P(1)$ is max.

$$P(x) = x^4 + ax^3 + bx^2 + cx + d.$$

$$P'(x) = 4x^3 + 3ax^2 + 2bx + c.$$

$x = 0$ is the root of $P'(x) \Rightarrow P'(0) = 0$.

$\Rightarrow c = 0$.

$$P'(x) = 4x^3 + 3ax^2 + 2bx.$$

$P(-1) < P(1)$

$\Rightarrow a > 0$.

$$P''(x) = 12x^2 + 6ax + 2b.$$

$$= 2(\underbrace{6x^2 + 3ax + b}_{D<0})$$

$(3a)^2 - 4(6)(b)$

$D < 0$.

$(3a)^2 - 4(6)(b) < 0$.

$9a^2 - 24b < 0$.

$9a^2 < 24b$.

$3a^2 < 8b$. $\Rightarrow 8b > 3a^2$.

$\Rightarrow b > \frac{3a^2}{8} > 0$.

$P(x) = x^4 + ax^3 + bx^2 + d$, $a, b > 0$.

$P'(x) > 0 \;\forall x$.

$\Rightarrow P(1)$ is max / Check $P(-1)$ is min. (HW).

Q. Let $f(x) = x|x|$ and $g(x) = \sin x$.

✓ S-I: $g \circ f$ is diff at $x = 0$ and its derivative is cont at $x = 0$.

✗ S-II: $g \circ f$ is twice diff at $x = 0$.

$$g\{f(x)\} = \sin(x|x|)$$

$$x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$= \begin{cases} \sin x^2, & x \geq 0 \\ -\sin x^2, & x < 0 \end{cases}$$

$$g\{f(x)\}' = \begin{cases} 2x \cos x^2, & x \geq 0 \\ -2x \cos x^2, & x < 0 \end{cases}$$

$LHL\big|_{x=0} = 0 = RHL\big|_{x=0}$.

$LHD\big|_{x=0} = 0 = RHD\big|_{x=0}$.

$$g\{f(x)\}'' = \begin{cases} 2\cos x^2 - 4x^2 \sin x^2, & x \geq 0 \\ -2\cos x^2 + 4x^2 \sin x^2, & x < 0 \end{cases}$$

Not diff at $x=0$.

Q. If $x_1, x_2, \ldots, x_n$ are the real roots of $x^n + ax + b = 0$, then value of: $(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n) = ?$

(a) $nx_1 + b$ (b) nx_1^{n-1} (c) $nx_1^{n-1} + a$ (d) None.

$$x^n + ax + b = (x-x_1)(x-x_2)\ldots(x-x_n)$$

$$\left[\frac{0}{0}\right] \frac{x^n + ax + b}{x - x_1} = (x-x_2)(x-x_3)\ldots(x-x_n)$$

Put $x = x_1$ on RHS: $(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)$

$$\therefore \lim_{x\to x_1} \frac{x^n + ax + b}{x - x_1} \quad \left[\frac{0}{0}\right]$$

L·Hopital: $\displaystyle\lim_{x\to x_1} \frac{nx^{n-1} + a}{1} = nx_1^{n-1} + a$

Q. Let $f: \mathbb{R}\to\mathbb{R}$ be a continuous fn defined by: $f(x) = \dfrac{1}{e^x + 2e^{-x}}$

S-I: $f(c) = \frac{1}{3}$, for some $c \in \mathbb{R}$. [Put $x=0$].

S-II: $0 < f(x) < \frac{1}{2\sqrt{2}}$ $\forall x \in \mathbb{R}$.

$\hookrightarrow$ Max value of $f(x)$.

$$f(x) = \frac{1}{e^x + 2e^{-x}} > 0.$$