GROUP THEORY SESSION 30 /11/2023

Thursday, November 30, 2023 7:35 PM

Quotient Groups

G . N subgroup of G

G/N with binary composition defined by

(Na)(Nb) = Nab

↳ GROUP of G by N

{1, 2, 3, 4} {1, 2}

Z ⊳ nZ normal subgroup

Z/3Z

0 + nZ
1 + nZ
2 + nZ
⋮
(n − 1) + nZ

order (3)

Order (n)

N → 1, 2, 3, 4, ...

3N → 3, 6, 9, 12 . . . .

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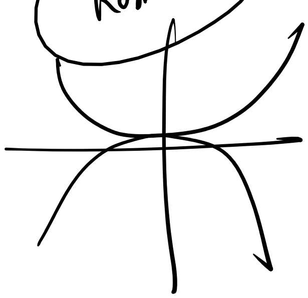
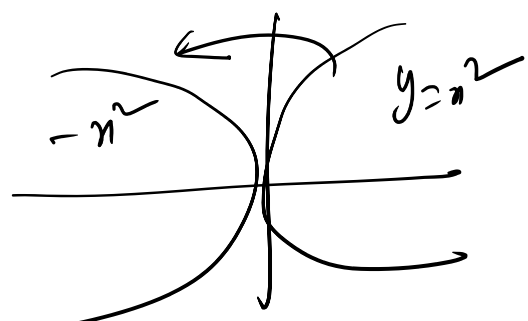
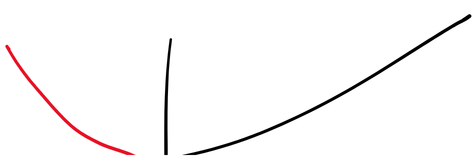
Dihedral Group

Group of symmetries of a regular polygon

Rotation or Reflections

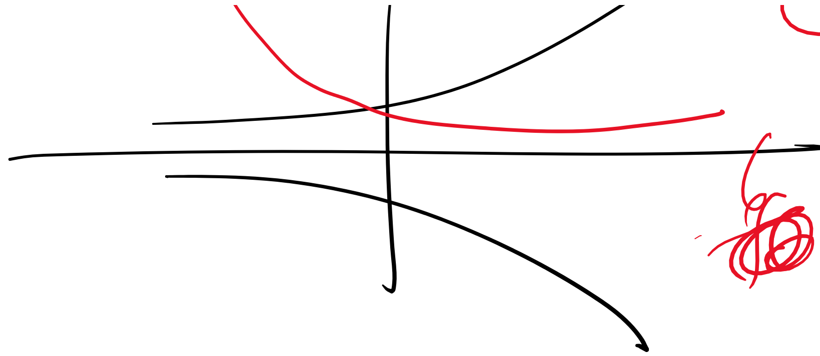






f(x) = f(−x)

f(x) ⇒ −f(x)



$f(x) \Rightarrow -f(x)$

Ist → working

2nd → ...

photography

Permutation Group

Permutation on a set S

1-1 mapping from S onto itself

HF ⟷ GF

10 km

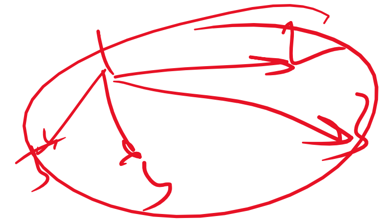
5 km

Parents → Neighbourhood

$y + x + x_1 + x_2 + x_3 + x_4 + f(x)$

Car

Cyclic form



Questions & Solutions from Past year

① If $Z(G)$ denotes centre of the Group G, $\frac{G}{Z(G)}$ can't be ... the Quotient Group ... ??

(1) If Z(G) ... Z(G)

then order of the Quotient ...

4 | 6 | 15 | 25 ??

Ans: If G/Z(G) is cyclic then G is Abelian

∴ If |G/Z(G)| = 15 then cyclic (As every group of order 15 is cyclic)

(order)

G is abelian ⇒ so, G = Z(G)

|G/Z(G)| = 1 not possible

Q. Q/Z → Additive group of Rational numbers..

Then order of the element $\left(\frac{2}{3}+Z\right)$ → (3)

Ans: Q/Z of Rational numbers

Q/Z is Z

Ans, as $\left(\frac{2}{3}+Z\right) + \left(\frac{2}{3}+Z\right) + \left(\frac{2}{3}+Z\right)$

$= \frac{2}{3} + \left(\frac{2}{3}+\frac{2}{3}+Z\right) = 2+Z = Z$

order of $\left(\frac{2}{3}+Z\right)$ ~~(2)~~ 3

Q. H is Group of 2×2 Invertible matrices Z_5

order of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in H ?

3/4/6/15 ~~7/5~~

(Z_5)

$\begin{bmatrix} 2 & 3 \end{bmatrix}$ ...

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}$$

(15)

$$A^3 = \underset{A}{\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}}\underset{A^2}{\begin{bmatrix} 2 & 2 \\ 4 & 2 \end{bmatrix}} = \begin{bmatrix} 16 & 10 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A^3 = I$ order (3)

$A^n = I$ order (n)

TIFR

Q GL_3 → General Linear (3)

$GL_3(\mathbb{C})$

NBHM 2021

IS $H = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C}$ as SG of GL_3 ??

Let $A = \begin{bmatrix} 1 & \alpha_1 & \beta_1 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & \alpha_2 & \beta_2 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix}$

$$B^{-1} = \begin{bmatrix} 1 & \alpha_2 & \alpha_2\gamma_2 - \beta_2 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AB^{-1} = \begin{bmatrix} 1 & \alpha_1 & \beta_1 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \alpha_2 & \alpha_2\gamma_2 - \beta_2 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \alpha_1 + \alpha_2 & \alpha_2\gamma_2 - \beta_2 + \alpha_1\gamma_2 + \beta_1 \\ & & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha_1+\alpha_2 & \alpha_2\gamma_2 - \beta_2 + \alpha_1\gamma_2 + \beta_1 \\ 0 & 1 & \gamma_1+\gamma_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$$

$AB^{-1} \in H$

H is a subgroup of GL_3

Q For α for which

$G = \{\alpha, 1, 3, 9, 19, 27\}$ is a cyclic group

under $\times$ modulo 56

~~56~~ 5 / 15 / (25) / 35 ??

Ans: Power of 3 generates the group

$3^1 \equiv 3 \pmod{56}$

$3^2 \equiv 9 \pmod{56}$

$3^3 \equiv 27 \pmod{56}$

$3^4 \equiv 25 \pmod{56}$

$3^5 \equiv 19 \pmod{56}$

$3^6 \equiv 1 \pmod{56}$

$\alpha = 25$

JAM-2015

Q. $5^{2015} \equiv n \pmod{11}$ $\quad n = ?$ ①

$\frac{5^{2015}}{11} \to R \to$ ①

$n \in \{0, 1, \ldots, 10\}$

Fermat's theorem. $a^{\phi(p)} \equiv 1 \pmod{p}$ $\quad p =$ prime

$\gcd(a, p) = 1$

$5^{10} \equiv 1 \pmod{11}$

$(5^{10})^{201} \equiv (1)^{201} \pmod{11}$ — ①

$5^{2010} \equiv 1 \pmod{11}$

$5^2 = 25 \equiv 3 \pmod{11}$

$(5^2)^2 \equiv 9 \pmod{11}$

$5^4 \equiv 9 \pmod{11}$

$5^5 \equiv 45 \pmod{11}$

$5^5 \equiv 1 \pmod{11}$ — 41

$5^{2010} \cdot 5^5 \equiv 1 \cdot 1 \pmod{11}$

$5^{2015} \equiv 1 \pmod{11}$

2023 Not a prime

$\frac{2023}{7} \to 289$

$7 \times 17 \times 17$

Q ** M.Math 2010

G non-abelian Group. $\alpha \in G$ have order (4)

$\beta \in G$ order (3)

Then order of the element $\alpha\beta$ in G ?

Ans: Let $G = S_7$ Let $\alpha = (1\,2\,3\,4)$

$\beta = (5\,6\,7)$

$o(\alpha) = 4$ $o(\beta) = 3$

$\alpha\beta = (1\,2\,3\,4)(5\,6\,7)$

$o(\alpha\beta) = \text{lcm}(o(\alpha), o(\beta)) = \text{lcm}(4,3) = 12$

Matrix

$\alpha = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\beta = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$\therefore \alpha\beta = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$o(\alpha\beta) =$ infinite $\left[\therefore (\alpha\beta)^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix} \right]$

Q Find elements of order (2) in a Symmetric Group S_4 of all permutation of the

Symmetric Group S_4 of all per...

four symbols $\{1,2,3,4\}$

Ans: elements of order 2 in S_4 are 2 cycles and

pr 2 cycles

(12), (13), (14), (23), (24), (34), (12)(34),

(13)(24), (14)(23)

So, no of elements of order 2 in S_4 → (9)