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$\boxed{\frac{dy}{dx} + P(x)\cdot y = Q(x)}$ ✓ $\frac{dx}{dy} + P(y)\cdot x = Q(y)$.

Q. Solve: $y e^{y}\, dx = (y^3 + 2x e^{y})\, dy$ given $x = 0,\ y = 1$.

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y} \quad \text{---- (i)}$$

$$P(y) = -2/y \qquad Q(y) = y^2 e^{-y}.$$

$$I.F = e^{\int P\,dy} = e^{-2\int 1/y\, dy} = e^{-2\ln|y|} = \frac{1}{y^2}.$$

(i) × I.F: $\frac{1}{y^2}\left[\frac{dx}{dy} - \frac{2}{y}\cdot x\right] = \frac{1}{\cancel{y^2}}\cdot \cancel{y^2} e^{-y}$

$$\frac{d}{dy}\left[x\cdot\frac{1}{y^2}\right] = e^{-y}.$$

Int: $\int d\left[x\cdot\frac{1}{y^2}\right] = \int e^{-y}\, dy$.

$$x\cdot\frac{1}{y^2} = -e^{-y} + c.$$

$x = 0,\ y = 1.$ $\quad 0 = -e^{-1} + c \Rightarrow c = e^{-1}$

$$\boxed{\frac{x}{y^2} = -e^{-y} + \frac{1}{e}}$$

(V) Bernoulli's Equations (Eqns reducible to Lin Diff Eqn) [soln technique]

Form: $\boxed{\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n}$ ✓

$\Rightarrow \quad y^{-n}\cdot\frac{dy}{dx} + y^{1-n}\, P(x) = Q(x)$ [Divide by y^n]

Let $y^{1-n} = z$.

Diff w.r.t x: $\boxed{(1-n)\cdot y^{-n}\cdot\frac{dy}{dx} = \frac{dz}{dx}}$.

$-n \quad \ldots \quad 1 \quad dz$

$$\frac{dz}{dx} \quad \frac{dz}{dx}$$

$$y^{-n} \cdot \frac{dy}{dx} = \frac{1}{(1-n)} \cdot \frac{dz}{dx}$$

Replace: $\frac{1}{(1-n)} \cdot \frac{dz}{dx} + z \cdot P(x) = Q(x)$.

$\boxed{\frac{dy}{dx} + P \cdot y = Q}$

$$\boxed{\frac{dz}{dx} + P(1-n) \cdot z = Q(1-n)} \Rightarrow$$ Linear Diff Eqn in z, x.

$$P' = P(1-n) \qquad Q' = Q(1-n)$$

Q. Solve: $\frac{dy}{dx} = y - y^2 - \frac{2}{3} y$.

$$\frac{dy}{dx} = \frac{1}{3} \cdot y - y^2$$

$$\frac{dy}{dx} - \frac{1}{3} y = - y^2$$ ---- Bernoulli Equation.

$$\frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{1}{3} \cdot \frac{y}{y^2} = -1.$$

$$y^{-2} \frac{dy}{dx} - y^{-1} \cdot \left(\frac{1}{3}\right) = -1 \quad \text{---- (i)}$$

Let $y^{-1} = z$.

Diff: $(-1) \cdot y^{-2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$

$$y^{-2} \frac{dy}{dx} = - \frac{dz}{dx}.$$

Put in (i): $- \frac{dz}{dx} - z\left(\frac{1}{3}\right) = -1$

HW. $\frac{dz}{dx} + \frac{1}{3} z = 1$ --- [Linear Diff Eqn in z, x]

(VI) Exact Differential Equations:

Form: $M\,dx + N\,dy = 0$ ------ $(*)$

where $M = M(x, y)$ $N = N(x,y)$.

A differential eqn of the form $(*)$ is called exact if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Q. $(x^2y + 1)\,dx + \left(\frac{1}{2}y + \frac{1}{3}x^3\right)dy = 0$.

check if it is exact diff eqn.

$M = (x^2y + 1)$ $N = \frac{1}{2}y + \frac{1}{3}x^3$

$\dfrac{\partial M}{\partial y} = x^2$ $\dfrac{\partial N}{\partial x} = x^2$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ Exact.

Form: $\boxed{M\,dx + N\,dy = 0}$.

Q. $\boxed{y\,dx - x\,dy = 0}$ $x, y > 0$. Check if eqn is exact

$M = y$ $N = -x$.

$\dfrac{\partial M}{\partial y} = 1$ $\dfrac{\partial N}{\partial x} = -1$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ Eqn is not exact.

$(*)$ Transform eqn to make it exact.

(i) Multiply eqn: $\left(\frac{1}{x^2}\right)$:

$$y\,dx - x\,dy = 0.$$

$$\Rightarrow \frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0.$$

$M = \dfrac{y}{}$ $N = -\dfrac{1}{}$

x^2 $\qquad\qquad$ x.

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2} \Rightarrow \text{Exact}.$$

(ii) Multiply eqn $\left(\frac{1}{y^2}\right)$: $\quad \Big\}$ exact.

(iii) Multiply eqn $\left(\frac{1}{xy}\right)$:

Note: $\frac{1}{x^2}$ / $\frac{1}{y^2}$ / $\frac{1}{xy}$ are the I.F of the diff eqn $y\,dx - x\,dy = 0$. I.F need not be unique.

Q. Find the I.F of : $x \sin y\, dx + (x+1)\cos y\, dy = 0$ --- (i)

$$M = x \sin y \qquad N = (x+1)\cos y.$$

$$\frac{\partial M}{\partial y} = x \cos y \qquad \frac{\partial N}{\partial x} = \cos y \Rightarrow \text{Not exact}.$$

To make it exact, we multiply (i) by I.F, let I.F $= f(x)$.

(i) × I.F $\Rightarrow$ $\underbrace{f(x)\cdot x \sin y}\, dx + \underbrace{f(x)(x+1)\cdot\cos y}\, dy = 0$. — (ii)

$$M' = f(x)\cdot x \sin y \qquad N' = f(x)\cdot(x+1)\cos y.$$

∴ For (ii) to be exact, $\boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$ --- (HW)