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(a) Probability mass function (pmf).

↓ probability distribution of a discrete r.v.

def:

For a discrete random variable X, there exists a function $f(x)$ such that

$$f(x) = p(X = x), \quad x \text{ is any value of } X$$

Conditions for X to follow pmf.

(i) $f(x) \geq 0$ for all value of x

(ii) $\sum_x f(x) = 1$

Also: Mean, $\boxed{E(x) = \sum_x x \cdot f(x)}$

and $\text{Var}(x) = E[x - E(x)]^2$

or $\boxed{\text{Var}(x) = E(x^2) - [E(x)]^2}$

where $\boxed{E(x^2) = \sum_x x^2 \cdot f(x)}$

Q Can the function

$$f(x) = \frac{1}{4} \quad \text{for } x = -1$$

$$= \frac{1}{4} \quad \text{for } x = 0$$

$$= \frac{1}{4} \quad \text{for} \quad x=0$$
$$= 1/2 \quad \text{for} \quad x=1$$
$$= 0 \quad \text{elsewhere}$$

be regarded as a pmf of some discrete r.v? If yes, then find Expectation and variance of the discrete r.v X.

<u>Soln</u> Here, $f(x) \geq 0$ for all values of x

an $\sum_x f(x) = f(-1) + f(0) + f(1)$
$$= 1/4 + 1/4 + 1/2$$
$$= 2/4 + 1/2 = 1$$

$$\therefore \sum_x f(x) = 1$$

Since both the conditions are satisfied $\therefore f(x)$ can be regarded as a p.m.f for a discrete r.v X.

$$E(x) = \sum_x x \cdot f(x)$$
$$= -1 \times f(-1) + 0 \times f(0) + 1 \times f(1)$$
$$= -1 \times 1/4 + 0 + 1 \times 1/2$$
$$= \frac{-1}{4} + \frac{1}{2} = \frac{2}{8}$$

$$= \frac{-1}{4} + \frac{1}{2} = \frac{2}{8}$$

$$= 1/4 \text{ (ans)}.$$

$$V(x) = E(x^2) - E(x)^2$$

$$E(x^2) = \sum x^2 \cdot f(x) = (-1)^2 \times f(-1) + 0^2 f(0) + 1^2 f(1)$$

$$= 1 \times 1/4 + 0 + 1 \times 1/2$$

$$= 1/4 + 1/2$$

$$= 3/4$$

$$\therefore V(x) = 3/4 - (1/4)^2$$

$$= 3/4 - 1/16 = \frac{11}{16} \text{ (ans)}.$$

Q2 The probability mass function $f(x)$ of a random variable X is zero, except at the point $(x = 0, 1, 2)$

and $f(0) = c$ ✓

$f(1) = 2c - 3c^2$ ✓

$f(2) = 4c - 1$ ✓

~~$f(0) = 2 > 1$~~
~~$f(1) = 4 - 12 = -8 < 0$~~
~~$f(2) = 7 > 1$~~

(i) Determine the value of c

(ii) Find $P(X > 0 / X < 2)$

(iii) Find $E(x)$ and $V(x)$.

(i) Since $f(x)$ is a p.m.f

$\therefore f(x) \geq 0$ for all values of x.

and $\sum_x f(x) = 1$

or, $f(0) + f(1) + f(2) = 1$

or, $c + 2c - 3c^2 + 4c - 1 = 1$

or $3c^2 - 7c + 2 = 0$

or $3c^2 - 6c - c + 2$

or $3c(c-2) - (c-2) = 0$

or $(3c-1)(c-2) = 0$

$\therefore 3c - 1 = 0$ and $c - 2 = 0$

or, $c = 1/3$ or $c = 2$

Since at $c = 2$, $f(0) = 2 > 1$

$\therefore c \neq 2$

Hence $c = 1/3$

$\therefore f(0) = 1/3$

$f(1) = 2 \times 1/3 - 3 \times (1/3)^2$

$= 2/3 - \frac{\cancel{3}}{\cancel{3}} 1/3$

$f(1) = 1/3$

and $f(2) = 4c - 1$

$= \frac{4}{3} - 1$

$= 1/3$

$\therefore f(0) = f(1) = f(2) = 1/3$.

Conditional ...)

$\therefore f(0) = f(1) = f(2) = 1/3$.

Conditional probability
$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$x = 0 \; 1 \; 2$

(ii) $P(x > 0 \mid x < 2)$

$$= \frac{P(x > 0 \cap x < 2)}{P(x < 2)}$$

$$= \frac{P(x = 1)}{P(x = 0) + P(x = 1)}$$

$$= \frac{f(1)}{f(0) + f(1)} = \frac{1/3}{1/3 + 1/3} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \times \frac{3}{2}$$

$= 1/2$ (ans).

(iii) $E(x) = \sum x f(x)$

$= 0 \times f(0) + 1 \times f(1) + 2 \times f(2)$

$=$

(iv) $v(x) = E(x^2) - (E(x))^2$

$= \sum x^2 . f(x) - (E(x))^2$

$=$

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Probability Density Function (Pdf)

Probability Density Function

↓ Probability Distribution of a continuous r.v. X

definition

For a continuous r.v x, there may exist a function $f(x)$ such that for $a \le b$,

$$\int_a^b f(x)\,dx = P(a \le x \le b)$$

Conditions: (i) $f(x) \geq 0$ for all values of x

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then x is a continuos r.v with a p.d.f.

Q3. Is the following a probability density function?

$$f(x) = \begin{cases} 2x, & 0 < x \le 1 \\ 4-2x, & 1 < x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Soln: Here for $0 < x \leq 1 \Rightarrow f(x) = 2x > 0$

and $1 < x \leq 2 \Rightarrow f(x) = 4 - 2x \geq 0$

$\therefore f(x) \geq 0$ for all values of x.

and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$= \int_0^1 (2x)\,dx + \int_1^2 (4-2x)\,dx$

$= 2\left[\frac{x^2}{2}\right]_0^1 + 4[x]_1^2 - 2\left[\frac{x^2}{2}\right]_1^2$

$= 1 + 4(1) - [4-1]$

$= 1 + 4 - 3$

$\int f(x)\,dx = 2 \neq 1$

$\therefore$ $f(x)$ is not a probability density function.

Q4: A continuous r.v X has a density function given by

p.df $\rightarrow$ $f(x) = \frac{1}{2} - ax, \quad 0 \leq x \leq 4$

$= 0$ elsewhere

Find (i) value of 'a'

(ii) $P(1 < X < 2)$ (iii) $P(2X + 3 > 5)$

(iv) $E(x)$

Solution

Since $f(x)$ is a p.d.f

$\therefore f(x) \geq 0$ for $0 \leq x \leq 4$

and $\int_0^4 f(x)\,dx = 1$

or, $\int_0^4 \left[\frac{1}{2} - ax\right] dx = 1$

or, $\frac{1}{2}(x)_0^4 - a\left[\frac{x^2}{2}\right]_0^4 = 1$

or $\frac{4}{2} - \frac{a}{2}[16] = 1$

or $2 - 8a = 1$

or $8a = 1$

or $a = \frac{1}{8}$

$$\therefore \begin{cases} f(x) = \frac{1}{2} - \frac{1}{8}x & \text{for } 0 \leq x \leq 4 \\ \quad\;\; = 0 & \text{for elsewhere} \end{cases}$$

(ii) $P(1 < x < 2) = \int_1^2 f(x)\,dx$

$= \int_1^2 (\frac{1}{2} - \frac{1}{8}x)\,dx$

$$= \int_{1}^{2} \left(\frac{1}{2} - \frac{1}{8}x\right) dx$$

$$= \frac{1}{2}[x]_1^2 - \frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} - \frac{1}{8}\left[\frac{2^2-1^2}{2}\right]$$

$$= \frac{1}{2} - \frac{5}{16} = \frac{6}{32}$$

$$= \frac{3}{16} \quad \text{(ans)}$$

(iii) $P(2x+3>5)$

$= P(2x>2)$

$= P(x>1)$

$$= \int_{1}^{4} f(x)\,dx = \int_{1}^{4} \left(\frac{1}{2} - \frac{1}{8}x\right) dx$$

$$= \frac{9}{16}$$

(iv) $$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_0^4 x \cdot \left[\frac{1}{2} - \frac{1}{8} x \right] dx$$

$$= \int_0^4 \left\{ \left[\frac{x}{2} - \frac{1}{8} x^2 \right] dx \right\}$$

$$= \left[\frac{x^2}{4} \right]_0^4 - \frac{1}{8} \left[\frac{x^3}{3} \right]_0^4$$

$$= \frac{4^2 - 0^2}{4} - \frac{1}{8 \times 3} \left[4^3 - 0^3 \right]$$

$$= \frac{\cancel{4^2}}{\cancel{4}} - \frac{4^3}{24}$$

$$= 4 - \frac{\cancel{16}\, 8}{\cancel{6}\, 3} = \frac{4}{3} \text{ (ans)}$$