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Recap: 2 r.s from 2 independent normal populations.

Sample 1 [n_1 obs]: $X_{11}, X_{12}, \ldots, X_{1n_1}$. $\Rightarrow X_1 \sim N(\mu_1, \sigma_1^2)$.

Sample 2 [n_2 obs]: $X_{21}, X_{22}, \ldots, X_{2n_2}$ $\Rightarrow X_2 \sim N(\mu_2, \sigma_2^2)$.

To test: $H_0: \mu_1 - \mu_2 = 0$. vs $H_{1A}: \mu_1 - \mu_2 > 0$.

$H_{1B}: \mu_1 - \mu_2 < 0$.

$H_{1C}: \mu_1 - \mu_2 \neq 0$.

Case I: σ_1^2, σ_2^2 are known. [already done!]

Case II: σ_1^2, σ_2^2 are unknown. ✓

Assume that variances are equal. $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (say)

Recall: From case I:
$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

If $\sigma_1^2 = \sigma_2^2$, then
$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{\left\{\frac{1}{n_1} + \frac{1}{n_2}\right\}}}$$

We know, $E(s_1'^2) = \sigma_1^2$, $\quad s_1'^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(x_{1i} - \bar{x}_1)^2$.

$E(s_2'^2) = \sigma_2^2$, $\quad s_2'^2 = \frac{1}{n_2 - 1}\sum_{j=1}^{n_2}(x_{2j} - \bar{x}_2)^2$

Now, $\hat{\sigma}^2$ will be estimated by the pooled estimator:

$$s_p^2 = \frac{(n_1-1)s_1'^2 + (n_2-1)s_2'^2}{(n_1-1) + (n_2-1)}$$

$= (n_1-1)s_1'^2 + (n_2 - 1)s_2'^2$

$(n_1 - 1) + (n_2 - 1)$

$$= \frac{(n_1-1)s_1'^2 + (n_2-1)s_2'^2}{(n_1+n_2-2)}$$

$\therefore$ New test-statistic: $T = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{\left\{\frac{1}{n_1} + \frac{1}{n_2}\right\}}} \overset{H_0}{\sim} t_{n_1+n_2-2}$.

Testing Rule:

(i) $H_0: \mu_1 - \mu_2 = 0$ vs $H_{1A}: \mu_1 - \mu_2 > 0$.

Reject H_0 at $\alpha\%$ L.O.S if $T_{obs} > t_{\alpha;(n_1+n_2-2)}$

(ii) $H_0: \mu_1 - \mu_2 = 0$ vs $H_{1B}: \mu_1 - \mu_2 < 0$

Reject H_0 at $\alpha\%$ L.O.S if $T_{obs} < -t_{\alpha;(n_1+n_2-2)}$

(iii) $H_0: \mu_1 - \mu_2 = 0$ vs $H_{1C}: \mu_1 - \mu_2 \neq 0$

Reject H_0 at $\alpha\%$ L.O.S if $|T_{obs}| > t_{\alpha/2;(n_1+n_2-2)}$

(IV) Testing for Population Variances:

(x) Case I: μ_1 and μ_2 are known [HW]

Case II: μ_1 and μ_2 are unknown.

To test: $H_0: \sigma_1^2 = \sigma_2^2$ vs $H_{1A}: \sigma_1^2 > \sigma_2^2 \Rightarrow \dfrac{\sigma_1^2}{\sigma_2^2} > 1$

$\hookrightarrow \dfrac{\sigma_1^2}{\sigma_2^2} = 1$.

$H_{1B}: \sigma_1^2 < \sigma_2^2 \Rightarrow \dfrac{\sigma_1^2}{\sigma_2^2} < 1$

$H_{1C}: \sigma_1^2 \neq \sigma_2^2 \Rightarrow \dfrac{\sigma_1^2}{\sigma_2^2} \neq 1$.

$\therefore$ We know, $E(s_1'^2) = \sigma_1^2$ $\quad s_1'^2 = \dfrac{1}{n} \sum^{n_1} (x_{1i} - \bar{x}_{1\cdot})^2$

$\therefore$ We know, $E(s_1'^2) = \sigma_1^2 \qquad s_1'^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_{1i}-\bar{x}_1)^2$

$E(s_2'^2) = \sigma_2^2 \qquad s_2'^2 = \frac{1}{n_2-1}\sum_{j=1}^{n_2}(x_{2j}-\bar{x}_2)^2$

$\therefore$ Test statistic: $T = \frac{s_1'^2}{s_2'^2} \overset{H_0}{\sim} F_{(n_1-1),(n_2-1)}$

Check: $T = \dfrac{\frac{1}{n_1-1}\sum(x_{1i}-\bar{x}_1)^2/\sigma_1^2}{\frac{1}{n_2-1}\sum(x_{2j}-\bar{x}_2)^2/\sigma_2^2} = \dfrac{\frac{1}{n_1-1}\underbrace{\sum\left(\frac{x_{1i}-\bar{x}_1}{\sigma_1}\right)^2}_{\chi^2_{(n_1-1)}}}{\frac{1}{n_2-1}\underbrace{\sum\left(\frac{x_{2j}-\bar{x}_2}{\sigma_2}\right)^2}_{\chi^2_{(n_2-1)}}}$

Form: $\dfrac{\chi^2_{(n_1-1)}/(n_1-1)}{\chi^2_{(n_2-1)}/(n_2-1)} \sim F_{n_1-1,\,n_2-1}$

and $\chi^2_{(n_1-1)}$, $\chi^2_{(n_2-1)}$ are independent

Testing Rule:

(i) $H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1$ vs $H_{1A}: \frac{\sigma_1^2}{\sigma_2^2} > 1$.

H_0 at $\alpha\%$ L.O.S if $T_{obs} > F_{\alpha;(n_1-1),(n_2-1)}$

(ii) $H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1$ vs $H_{1B}: \frac{\sigma_1^2}{\sigma_2^2} < 1$

H_0 rejected at $\alpha\%$ L.O.S if, $T_{obs} < F_{(1-\alpha);(n_1-1),(n_2-1)}$

(iii) $H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1$ vs $H_{1C}: \frac{\sigma_1^2}{\sigma_2^2} \neq 1$.

H_0 rejected at $\alpha\%$ L.O.S if $T_{obs} > F_{\alpha/2;(n_1-1),(n_2-1)}$

or, T < F

H_0 rejected at $\alpha\%$ L.O.S if $T_{obs} > F_{\alpha/2;(n_1-1),(n_2-1)}$

or $T_{obs} < F_{(1-\alpha/2);(n_1-1),(n_2-1)}$

(V) Testing for Bivariate Normal Distribution

Suppose we have 2 r.v's (X,Y) jointly distributed as normal $\therefore (X,Y) \sim BN(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{XY})$

Note: (i) 2 samples from Independent Normal Poplns

$\Rightarrow$ Sampling units [from the popln] are different.

(ii) Jointly distributed r.v's $\Rightarrow$

Sampling units are the same.

Eg: pre-&-post experiment studies.

$\therefore$ Consider a random sample (X_i, Y_i) of size n, $i=1,2,\cdots,n$

Define $d_i = X_i - Y_i$

and suppose μ_D and σ_D^2 be the mean and variance of d_i

Sample mean of diff: $\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i$

Sample variance of diff: $s_d'^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2$

Case I: Testing for means:

To test: $H_0: \mu_X = \mu_Y$ vs $H_{1A}: \mu_X > \mu_Y \to \mu_X - \mu_Y > 0$.

$\downarrow$

$\boxed{\mu_X - \mu_Y} = 0$.

$\to \mu_D$ ( )

$H_{1B}: \mu_X < \mu_Y \to \mu_X - \mu_Y < 0$

$H_{1C}: \mu_X \neq \mu_Y \to \mu_X - \mu_Y \neq 0$.

To test: $H_0 : \mu_D = 0$ vs $H_{1A} : \mu_D > 0$.

$H_{1B} : \mu_D < 0$

$H_{1C} : \mu_D \neq 0$.